FINITE ELEMENT METHOD FOR OPTIMUM BEAM DESIGN

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THESIS

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OPTIMUM BEAM DESIGN

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ABSTRACT

A finite element method for structural optimization of a prismatic beam of a homogeneous, isotropic material is developed. The beam has a rectangular cross-section of constant fixed height, a fixed length, and a fixed volume. Structural optimum is defined as that shape which allows a maximum load within the elastic range.

A computer program is developed to solve the resulting system of equations and various example problems are solved. Comparison is made with exact optimum beam designs where possible.

The finite element model is able to solve problems with any boundary conditions and types of loading that are consistent with the number of elements selected.

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LIST OF SYMBOLS

T	Potential energy functional
T^*	Augmented potential energy functional
S	Isoperimetric constraint
λ	Lagrange constant multiplier
L	Beam length
V_o	Beam volume
C	A constant
n	Exponent of area in inertia term; number of elements
$A(x)$	Cross-sectional area
x	Global beam longitudinal coordinate axis
y	Global beam vertical axis
v	Beam displacement in the y direction
v'	Beam slope, dv/dx
v''	Beam curvature, d^2v/dx^2
P_o	Load intensity
E	Modulus of elasticity
K_1, K_2	Constants of integration
I	Moment of inertia
S_y	Yield stress
M	Moment
R	Distance of beam fiber to neutral axis
b	Beam base
h	Beam height
Δ	Element length

θ	Element slope at ends
$N(x)$	Shape function for displacement
{ }	Column vector
< >	Row vector
U	Strain energy of bending
W	Potential energy of external forces
u_i	Strain energy of bending for i^{th} element
w_i	Potential energy of external forces for i^{th} element
$[k^*]$	Modified element stiffness matrix
k^*_{ij}	Component of $[k^*]$, i^{th} row, j^{th} column
$<D>_i$	Consistent load distribution vector for i^{th} element
$<\bar{D}>$	Global consistent load distribution vector
$\{v\}_i$	Element displacement/slope vector (local coordinates)
$\{\bar{v}_i\}$	Element displacement/slope vector (global coordinates)

I. INTRODUCTION

Optimization has a history as old as the universe itself. The planets have taken their optimum position in the solar system, the solar system in the galaxy, etc. The whole history of evolution is one of gradual optimization. Man has used optimization in one form or another since the earliest of times. Individual man was weak with respect to his hostile environment. He learned to become a social animal so that as a group he was strong - optimization. In wars between groups spears defeated rocks, arrows defeated spears, and so on to the present day. All were essentially optimization.

The early Egyptians formalized some optimization techniques in their development of geometry and trigonometry. The development of the Calculus enabled us to find optimum function values. The Calculus of Variations made possible the optimization of functionals, the optimum function from a function space.

In structural optimization, however, oftentimes the Calculus of Variations produces nonlinear differential equations that cannot be solved in closed form and the numerical methods of solution are difficult. If these differential equations could be transformed into algebraic equations, a solution might be more readily obtained.

The finite element method has been applied to the theories of elasticity [1], [2] and mechanical vibrations [3], as well as to other fields, to resolve such problems. It has been used in the optimization of truss networks but as yet not in actual beam optimization. Since the method is essentially variational in nature, it should have an application to structural optimization theory. This suggests the desirability for research in this field.

Since this is a first attempt at a formulation of this type, we will restrict ourselves to a specific type of problem; the structural optimization of a prismatic beam of one material which is homogeneous and isotropic. This beam has a rectangular cross-section of uniform fixed height, and has a given volume and length. The structural optimum in this case is the shape required to maximize the load within the elastic range.

A variational formulation is first presented to establish the optimization technique for isoperimetric problems. The finite element method is applied to the problem using a parallel approach. A model is developed and then checked for validity by solving various applicable problems.

II. VARIATIONAL FORMULATION OF BEAM OPTIMIZATION PROBLEMS

A. DEFINITION OF THE PROBLEM

Salinas [4] shows that the problem of structural optimization of a beam, under a fixed volume constraint, may be solved by using calculus of variations techniques. He shows that operation with the techniques for isoperimetric problems [5] on the augmented potential energy functional, $T^* = T - \lambda S$, produces equations which describe the structurally optimum beam. T is the potential energy of the beam and S is the isoperimetric constraint. The quantity λ is a constant Lagrange multiplier which is actually a measure of the strength of the beam.

We shall consider the problem of structural optimization of a simply supported beam of length, L, and volume, V_o. The applied load is uniform, P_o (force/unit length). The cross-section of the beam is such that the moment of inertia may be given as $I = CA^n$ where C and n are constants. Gravity forces and shear effects are assumed negligible and the usual beam assumptions [6] are used.

i) small deflection theory

ii) uniaxial stress state

Figure 1 graphically depicts the beam and defines the x and y axes. We seek the shape of the beam to give a maximum load intensity subject to the constant volume constraint.

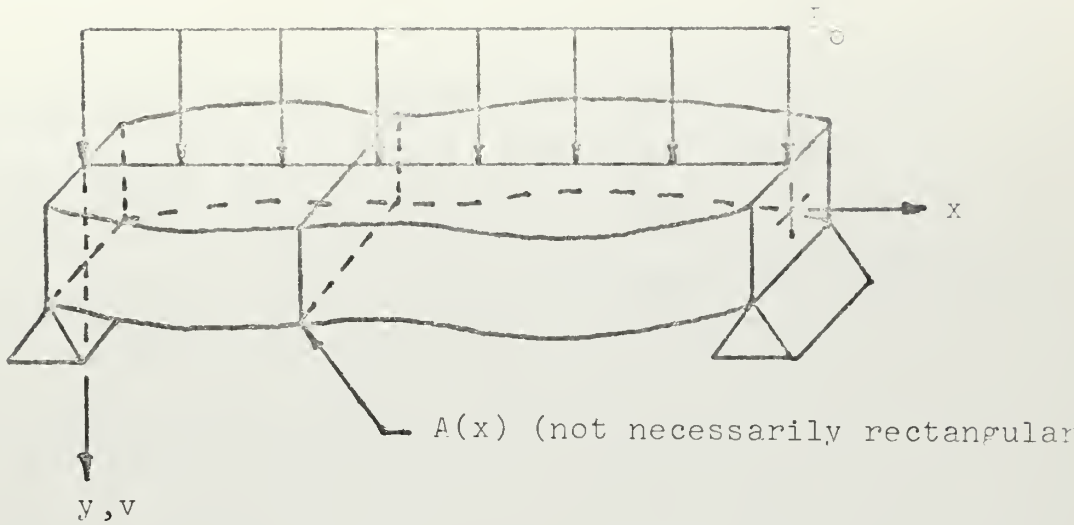


Figure 1. Simply Supported Beam.

B. THE VARIATIONAL FORMULATION

We first form the potential energy functional from strength of materials considerations. The displacement of the beam is denoted by v and the primes indicate differentiation with respect to x.

$$T = \int_0^L \left[\frac{EI}{2} (v'')^2 - P_o v\right] dx$$

The fixed volume constraint may be written as $\int_0^L A(x)dx=V_o$ thus the constraint augmentation becomes:

$$\lambda S = \lambda \int_0^L A(x)\, dx$$

Noting that I(x) may be denoted as $CA(x)^n$, the augmented potential energy functional is:

$$T^* = \int_0^L \left[\frac{ECA(x)^n}{2} (v'')^2 - P_o v - \lambda A(x)\right] dx$$

where v and v" are v(x) and v"(x) the displacement and curvature of the beam respectively.

In order to extremize the above functional, we first take the variation [7] of the functional with respect to displacement, v, and set this equal to zero. The result is the differential equation of equilibrium. This variation yields:

$$(EAC^n v'')'' - P_o = 0$$

which may be written:

$$(A^n v'')'' = P_o/EC \ .$$

Integrating this equation twice with respect to x gives:

$$A^n v'' = P_o/EX \left(\frac{x^2}{2} + K_1 x + K_2 \right)$$

The constants of integration, K_1 and K_2, are obtained from the boundary conditions. For a simply supported beam, the end moments are zero, or:

(a) $A^n(0)v''(0) = 0$

(b) $A^n(L)v''(L) = 0$

Boundary conditions (a) and (b) yield $K_2 = 0$, $K_1 = -L/2$ respectively.

Thus we obtain the differential equation of equilibrium for the simply supported beam.

$$A^n v'' = \frac{P_o}{2EC} (x^2 - Lx) \tag{II-1}$$

Taking the variation of the functional with respect to area, A(x), and setting this equal to zero, gives the differential equation which we refer to as the optimality

condition, For our example the optimality condition is:

$$\frac{nEC}{2} (A)^{n-1} (v'')^2 - \lambda = 0$$

Solving for v" yields:

$$v'' = \sqrt{\frac{2\lambda}{nEC}}\ A^{-\left(\frac{n-1}{2}\right)} \qquad \text{(II-2)}$$

$$A^{\frac{n-1}{2}} v'' = \sqrt{\frac{2\lambda}{nEC}} \qquad \text{(II-2a)}$$

Equation (II-2a) shows that, for an optimum beam, the curvature is constant only for n = 1. Otherwise the product of a power of the cross-sectional area times the curvature is constant.

Multiplying both sides of equation (II-2) by A^n and substituting into equation (II-1), the development continues as:

$$A^n v'' = A^{\frac{n+1}{2}} \sqrt{\frac{2\lambda}{nEC}} \qquad \text{(II-3)}$$

From equation (II-1):

$$A^{\frac{n+1}{2}} \sqrt{\frac{2\lambda}{nEX}} = \frac{P_o}{2EC} (x^2-Lx)$$

Hence,

$$A^{n+1} = \frac{nP_o^2}{8EC\lambda} (x^2-Lx)^2$$

Since $(x^2-Lx)^2 = (Lx-x^2)^2$ it is convenient in further development to use the latter expression. Substituting this into the previous equation yeilds:

$$A^{n+1} = \frac{nP_o^2}{8EC\lambda}(Lx-x^2)^2$$

Taking the $(n+1)^{th}$ root of both sides:

$$A = \left[\frac{nP_o^2}{8EC\lambda}(Lx-x^2)^2\right]^{\frac{1}{n+1}} \tag{II-4}$$

Introducing the constraint equation and solving for λ we obtain:

$$\int_0^L A(x)dx = \int_0^L \left[\frac{nP_o^2}{8EC\lambda}(Lx-x^2)^2\right]^{\frac{1}{n+1}} dx = V_o$$

But since λ is a constant we obtain,

$$\lambda = \left[\left(\frac{nP_o^2}{8EC}\right)^{\frac{1}{n+1}}\left(\frac{1}{V_o}\right)\int_0^L (Lx-x^2)^{\frac{2}{n+1}}dx\right]^{n+1} \tag{II-5}$$

Rearranging (II-5) and solving for P_o we obtain the equation defining the maximum load intensity

$$P_o = \frac{\sqrt{\frac{8EC\lambda}{n}}\,(V_o)^{\frac{n+1}{2}}}{\left[\int_0^L (Lx-x^2)^{\frac{2}{n+1}}dx\right]^{\frac{n+1}{2}}} \tag{II-6}$$

Substituting (II-5) into (II-4) we obtain for the optimum area:

$$A = \frac{V_o(Lx-x^2)^{\frac{2}{n+1}}}{\int_0^L (Lx-x^2)^{\frac{2}{n+1}}dx} \tag{II-7}$$

It should be noted here that the integral is not defined in closed form for all n and therefore must be numerically approximated for some problems.

Summarizing the equations that define the solution to a given problem we have:

$$A = \frac{V_o (Lx-x^2)^{\frac{2}{n+1}}}{\int_0^L (Lx-x^2)^{\frac{2}{n+1}} dx} \qquad \text{(II-8)}$$

$$P_o = \frac{\sqrt{\frac{8EC\lambda}{n}}\,(V_o)^{\frac{n+1}{2}}}{\left[\int_0^L (Lx-x^2)^{\frac{2}{n+1}} dx\right]^{\frac{n+1}{2}}} \qquad \text{(II-9)}$$

The value of λ is determined as follows. From equation (II-2a):

$$v'' = \sqrt{\frac{2\lambda}{nECA^{n-1}}}$$

From strength of materials theory:

$$M_{max} = - E(Iv'')_{max}$$

If we define the yield stress of the beam material, S_y, as the maximum allowable stress, we obtain:

$$S_y = \frac{|M|_{max} R_{max}}{I}$$

where R_{max} is the vertical distance to the furthest fiber from the neutral axis. Thus:

$$S_y = \frac{E(Iv'')_{max}R_{max}}{I} = Ev''_{max}R_{max}$$

Substituting for v":

$$S_y = ER_{max}\sqrt{\frac{2\lambda}{ECA^{n-1}_{max}}}$$

$$S_y^2 = \frac{2\lambda E}{n}\left(\frac{R^2}{CA^{n-1}}\right)_{max}$$

$$\lambda = \frac{nS_y^2}{2E}\left(\frac{CA^{n-1}}{R^2}\right)_{max} \qquad \text{(II-10)}$$

It may be shown by example that the quantity $\frac{CA^{n-1}}{R^2}$ is independent of x and thus that λ is indeed a constant for a given type of cross-section.

The first example is a constant height rectangular cross-section of variable width. The height and width are h, and b(x) respectively.

$$I(x) = \frac{b(x)h^3}{12} = b(x)h \cdot \frac{h^2}{12}$$

$$A(x) = b(x)h$$

$$I(x) = \frac{h^2}{12} A(x)$$

$$\therefore \quad C = \frac{h^2}{12}, \quad n = 1, \quad R = h/2$$

$$\frac{CA^{n-1}}{R^2} = \frac{C}{R^2} = \frac{h^2/12}{h^2/4} = \frac{1}{3}$$

Thus for a rectangular cross-section as above, equation (II-10) becomes:

$$\lambda = \frac{S_y^2}{6E} \tag{II-11}$$

The second example is a circular cross-section of radius r(x).

$$I(x) = \frac{\pi r(x)^4}{4} = (\pi r(x)^2)^2 \cdot (\frac{1}{4\pi})$$

$$A(x) = \pi r(x)^2$$

$$I(x) = \frac{1}{4\pi} A(x)^2$$

$$\therefore \quad C = \frac{1}{4\pi}, \quad n = 2, \quad R = r$$

$$\frac{CA^{n-1}}{R^2} = \frac{CA}{R^2} = \left(\frac{1}{4\pi}\right) \cdot (\pi r^2) \cdot \left(\frac{1}{r^2}\right) = \frac{1}{4}$$

Thus for a circular cross-section, equation (II-10 becomes:

$$\lambda = \frac{S_y{}^2}{4E} \tag{II-12}$$

The third example is a rectangular cross-section of constant width, b, and variable height, h(x).

$$I(x) = \frac{bh(x)^3}{12} = b^3 h(x)^3 \cdot \frac{1}{12b^2}$$

$$A(x) = b \cdot h(x)$$

$$I(x) = \frac{1}{12b^2} A(x)^3$$

$$\therefore \quad C = \frac{1}{12b^2}\ , \quad n = 3, \quad R = h/2$$

$$\frac{CA^{n-1}}{R^2} = \frac{CA^2}{R^2} = \frac{1}{12b^2} \cdot (b^2h^2) \cdot \left(\frac{4}{h^2}\right) = \frac{1}{3}$$

Thus for a rectangular cross-section of constant width and variable height, equation (II-10) becomes:

$$\lambda = \frac{S_y^{\ 2}}{2E} \qquad \text{(II-13)}$$

Note that equations (II-11), (II-12), (II-13) depend only on cross section type and not on the manner of loading (independent of the term $\int_0^L P(x)v(x)dx$). They are also independent of the boundary conditions since they were derived from equations in which the boundary conditions had not been envoked. This observation will be used in Section III.

Equations (II-8), (II-9), and (II-10), when applied to a specific problem, yield the structurally optimum beam shape for maximum load intensity for beams which conform to the assumptions made at the beginning of this section. These beams must be simply supported and under a uniform load distribution.

Appendix A gives three examples of structural optimization problems solved with equations (II-8, 9, 10).

III. A FINITE ELEMENT MODEL FOR BEAM OPTIMIZATION

A. SPECIFICATIONS AND ASSUMPTIONS

This development parallels that of the continuous variational formulation of the preceding section. We consider prismatic beams of one material which is homogeneous and isotropic. The modulus of elasticity and yield stress of the material are E and S_y respectively. The cross-sectional area of the beam is rectangular with constant height, h, and variable width, b(x). The length and volume of the beam are denoted by L and V_o respectively.

The plane of loading is in the centroidal plane parallel to the y axis. Gravity forces and shear effects are assumed to be negligible. The usual beam assumptions [6] are taken:

i) small deflection theory

ii) uniaxial stress state

B. STATEMENT OF THE PROBLEM

As in Section II, we seek to optimize the shape of the beam (cross-sectional area as a function of position along the beam axis) to give a maximum load intensity subject to the constraint that the beam volume is fixed. The finite element method results in an approximate solution to this problem.

C. DEFINITION OF THE FINITE ELEMENT

The beam of Figure 1, without supports, is discretized into finite elements of equal length as shown in Figure 2. The elements are identified by numbering them sequentially from the origin. The x (centroidal) axis and y axis (positive downward) define the plane of loading and bending.

We consider a typical element, say the i^{th}, as shown in Figure 3. The displacements and slopes at the left and right ends of the element are denoted by $v_1, \theta_1, v_2, \theta_2$ respectively. The cross-sectional area is A_i. The local coordinate system x_i and y_i are defined for each element.

D. APPLICATION OF THE FINITE ELEMENT METHOD

In accordance with the finite element method [1], shape functions are assumed for the displacement of each element. Since a cubic polynomial function for v_i, the displacement over the i^{th} element, yields continuity of displacements and slopes at each element boundary, it is taken as the assumed displacement field. Thus the displacement over the i^{th} element may be written:

$$v(x_i) = \underset{1 \times 4}{\langle N(x_i) \rangle} \; \underset{4 \times 1}{\{v\}_i}$$

where $\langle N(x_i) \rangle$ is the cubic shape function vector as developed in Appendix B-1 and $\{v\}_i$ is the column vector $(v_1, \theta_1, v_2, \theta_2)_i$.

The elemental cross-sectional area, A_i, is assumed to be constant over the element to obtain a tractable mathematical model. This point is further discussed later in the development.

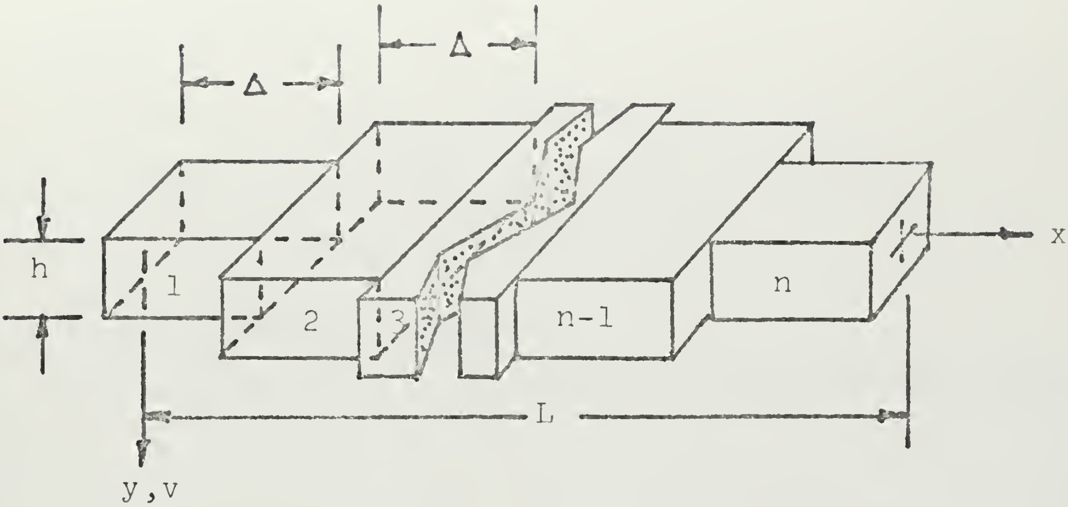


Figure 2. A Finite Element Beam.

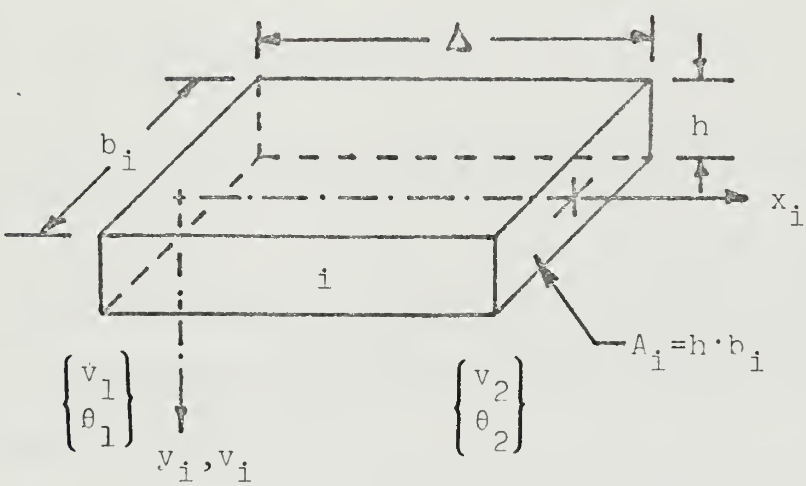


Figure 3. The Ith Finite Element.

Since the augmented potential energy functional, $T^* = U-W-\lambda S$, must be extremized, finite element formulations must be obtained for the:

i) strain energy of bending (U)

ii) potential energy of external force (W)

iii) isoperimetric constraint (S)

1. Strain Energy of Bending

The strain energy of bending of a beam may be given as:

$$U = \int_0^L \frac{EI}{2} (v'')^2 \, dx$$

The finite element assumption that each element is under uniaxial stress enables the strain energy in the i^{th} element, u_i, to be expressed as:

$$u_i = \int_0^{\Delta} \frac{EI_i}{2} (v_i'')^2 \, dx_i$$

where v_i'' denotes the second derivative of $v(x_i)$ with respect to x_i.

For a rectangular section of constant height, h, and width, b_i, the moment of inertia of any section of the element may be written as:

$$I_i = CA_i \qquad C = h^2/12$$

The strain energy in the i^{th} element is obtained as follows:

$$v(x_i) = \langle N(x_i) \rangle \{v\}_i$$

$$v''(x_i) = \langle N''(x_i)\rangle \{v\}_i$$

$$(v'')^2 = \{v\}_i^T \langle N''\rangle_i^T \langle N''\rangle_i \{v\}_i$$

$$u_i = \int_0^{\Delta} \frac{EC}{2} A_i \underset{1x4}{\{v\}_i^T} \underset{4x1}{\langle N''\rangle_i^T} \underset{1x4}{\langle N''\rangle_i} \underset{4x1}{\{v\}_i} dx_i$$

Since the product of the second derivative of the shape function vector, $\langle N_i''{}^T N_i''\rangle$, is the only function of x_i in the expression, the remaining terms may be brought outside the integral, and the integral may be evaluated. Thus.

$$u_i = \frac{EC}{2} A_i \{v\}_i^T \int_0^{\Delta} \langle N''\rangle_i^T \langle N''\rangle_i \, dx_i \{v\}_i$$

The integral is evaluated in Appendix B-2 and called [k*], a 4x4 matrix. Therefore:

$$u_i = \frac{EC}{2} A_i \underset{1x4}{\{v\}_i^T} \underset{4x4}{[k^*]} \underset{4x1}{\{v\}_i}$$

Here it may be noted that, if the A_i has a shape function vector associated with it other than a constant, the vector $\{v\}_i^T$ could not be removed from the integrand. Since the values for the vector $\{v\}_i^T$ are unknown, this vector would appear in the integrand between two shape function vectors, and therefore closed form integration is not possible.

When the term ECA_i is included in the integrand, the result of integration is commonly known as the element stiffness matrix. In the present formulation [k*] is called

the "modified element stiffness matrix" since the k^*_{ij} are measures of element stiffness.

Since strain energy is a scalar, the total strain energy in the beam is the sum of the individual element strain energies. Hence we have:

$$U = \sum_{i=1}^{n} u_i = \sum_{i=1}^{n} \frac{EC}{2} A_i \{v\}_i^T [k^*] \{v\}_i$$

where n is the number of elements.

2. <u>Potential Energy of External Forces</u>

The potential energy of external forces for the i^{th} element may be written as:

$$w_i = P_o \underset{1x4}{<D>_i} \underset{4x1}{\{v\}_i}$$

The vector $<D_i>$ is called the element consistent load distribution vector and is described in detail in Appendix B-3.

Since the potential energy is also a scalar, the total is the sum of the contributions of the individual elements. Therefore:

$$W = P_o \sum_{i=1}^{n} <D>_i \{v\}_i$$

3. <u>The Augmented Potential Energy Functional</u>

The constant volume constraint in finite element form is $\sum_{i=1}^{n} A_i \Delta = V_o$. In order to maximize the load intensity P_o we form the augmented potential energy functional,

$$T^* = \sum_{i=1}^{n} \frac{EC}{2} A_i \{v\}_i^T [k^*] \{v\}_i - \sum_{i=1}^{n} P_o <D>_i \{v\}_i - \lambda \sum_{i=1}^{n} A_i \Delta$$

E. THE FINITE ELEMENT VARIATIONAL PROBLEM

The extremization of the potential energy functional must be done globally, i.e., over the entire beam. A transformation is therefore made from the local coordinates $\{v\}_i$ to a global coordinate system $\{\bar{v}_i\}$. The transformation is made by numbering the components of $\{\bar{v}_i\}$ sequentially from the origin, where the first component is the displacement at the origin, the second is the slope at the origin, etc. Thus:

$$\bar{v}_i = v_1^1$$

$$\bar{v}_2 = \theta_1^1$$

$$\bar{v}_3 = v_2^1 = v_1^2$$

$$\bar{v}_4 = \theta_2^1 = \theta_1^2 \quad \ldots$$

Therefore the global representation of the local "displacement" vector $\{v\}_i$ becomes:

$$\{v\}_i = \begin{Bmatrix} \bar{v}_{2i-1} \\ \bar{v}_{2i} \\ \bar{v}_{2i+1} \\ \bar{v}_{21+2} \end{Bmatrix} = \{\bar{v}_i\}$$

In global coordinates the potential energy functional becomes:

$$T^* = \frac{EC}{2} \sum_{i=1}^{n} A_i \, \{\bar{v}_i\}^T \, [k^*] \, \{\bar{v}_i\} - P_o \sum_{i=1}^{n} \langle D\rangle_i \, \{\bar{v}_i\} - \lambda\Delta \sum_{i=1}^{n} A_i$$

The algebraic equations describing the structurally optimum beam shape and maximum permissible load intensity arise from taking the partial derivatives of the global augmented potential energy functional with respect to the state variables of individual slopes and displacements and also with respect to the control variables of the element areas and setting these equal to zero. The constraint equation is also imposed.

The following set of equations, as developed in Appendix B-4, results. The equations must be numbered as shown to facilitate enforcement of boundary conditions. This is described later in the development.

$$A_1 \sum_{j=1}^{4} k^*_{1j} \bar{v}_j - P_o \frac{\bar{D}_1}{EC} = 0 \qquad (1)$$

$$A_1 \sum_{j=1}^{4} k^*_{2j} \bar{v}_j - P_o \frac{\bar{D}_2}{EC} = 0 \qquad (2)$$

$$A_{\left(\frac{i-2}{2}\right)} \sum_{j=1}^{4} k^*_{3j} \bar{v}_{(j+i-4)} + A_{(i/2)} \sum_{j=1}^{4} k^*_{1j} \bar{v}_{(j+i-2)} - \frac{P_o \bar{D}_{(i-1)}}{EC} = 0 \qquad (i-1)$$

$$A_{\left(\frac{i-2}{2}\right)} \sum_{j=1}^{4} k^*_{4j} \bar{v}_{(j+i-4)} + A_{(i/2)} \sum_{j=1}^{4} k^*_{2j} \bar{v}_{(j+i-2)} - \frac{P_o \bar{D}_i}{EC} = 0 \qquad i = 4,6,\ldots 2n \qquad (i)$$

$$A_n \sum_{j=1}^{4} k^*_{3j} \bar{v}_{(j+2n-2)} - \frac{P_o \bar{D}_{(2n+1)}}{EC} = 0 \qquad (2n+1)$$

$$A_n \sum_{j=1}^{4} k^*_{4j} \, \bar{v}_{(j+2n-2)} - \frac{P_o \bar{D}_{(2n+2)}}{EC} = 0 \qquad (2n+2)$$

$$\sum_{i=1}^{4} \sum_{j=1}^{4} k^*_{ij} \bar{v}_{(j+2m-2)} \bar{v}_{(i+2m-2)} - \frac{2\lambda\Delta}{EC} = 0 \qquad (2n+2+m)$$

$$m = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} A_i - \frac{V_o}{\Delta} = 0 \qquad (3n+3)$$

where:

$\bar{D}_i$ = i^{th} component of the global consistent load distribution vector, $\langle \bar{D} \rangle_{1\times(2n+2)}$

λ = strength constant $S_y^2/6E$ as developed in Section II.

$\bar{v}_i$ = i^{th} component of the global displacement vector.

This is a set of 3n+3 nonlinear algebraic equations which, when solved simultaneously, results in:

i) values for n+1 nodal displacements and n+1 nodal slopes at maximum load intensity.

ii) the n optimum element areas

iii) the maximum load intensity, P_o

No boundary conditions are included in the preceding global formulation. Since the derivatives may only be taken with respect to variables, a fixed displacement or slope boundary condition requires the removal of the equation associated with the variation of T* with respect to that boundary condition. Its value must be substituted in the remaining equations. Boundary conditions must be consistent with the finite element model, i.e. they must occur at nodal points.

For example, if the j^{th} state variable, $\bar{v}_j$, is fixed, the j^{th} equation is eliminated from the set, and the value for $\bar{v}_j$ substituted in the remaining equations.

In this way, boundary conditions are included in the formulation for specific problems.

The set of equations with boundary conditions considered is the finite element model for the approximate structural optimization of a particular rectangular cross-section beam of uniform height and fixed volume.

IV. APPLICATION OF THE FINITE ELEMENT MODEL

In order to apply the model developed in the preceding section we must find some method for solving the system of nonlinear simultaneous algebraic equations. The size of the system for even a small number of elements requires the use of a computer algorithm for solution. The only such algorithm available to the author was a Fortran IV-G subroutine called SUBROUTINE NLNSYS. This subroutine is included in the computer program section of this report.

A. SUBROUTINE NLNSYS

This subroutine, in theory, solves simultaneously, by an iterative scheme developed by Brown [8], [9], [10], a system of nonlinear algebraic equations of any order. As it is dimensioned it is restricted to, at most, 100 equations.

NLNSYS requires as input a vector of starting values for all unknowns. This starting guess vector must be in the region necessary for convergence or the process will diverge. This region cannot be defined and different starting guesses may be necessary to obtain convergence. It should be noted that large systems of equations require large starting guess vectors. Since the probability of a component being outside the region of convergence increases with the number of components, the probability of a starting guess vector being within this range decreases with

increasing system size. This was found to be a difficulty that limited the size of the example problems to three elements.

NLNSYS also requires an external subroutine that specified the system of equations. Because the system of equations developed in Section III changes with the number of elements and boundary conditions of a problem, a general subroutine is desired that will generate the proper set of equations for a given problem. SUBROUTINE FCNLST, provided in the program section, accomplishes this task.

B. SUBROUTINE FCNLST

SUBROUTINE FCNLST generate a set of equations which are evaluated at a point provided by NLNSYS. NLNSYS selects only one equation at a time, thus the set of equations must be ordered in some manner.

SUBROUTINE FCNLST sequences the equations for a given number of elements as they are ordered in Section III. It evaluates these equations at the point specified by NLNSYS.

The equations derived from parameters that are fixed by boundary conditions are removed from the general set and NLNSYS operates on one of the equations in this reduced set.

Problem parameters are read into the program in SUBROUTINE FCNLST the first time that it is called by NLNSYS.

C. SUBROUTINE OPBEAM

SUBROUTINE OPBEAM, as provided in the program section, reads the initial guess vector into the program and provides

the output from the program. It permits an efficient means of trying multiple starting guess vectors for a given problem. It starts the solution to the problem by calling NLNSYS.

D. INSTRUCTIONS FOR PROGRAM USE

It is first necessary to define the problem to be solved. We specify:

1) the length of the beam (in) - BLENG
2) the volume (in^3) - VOL
3) the height (in) - HI
4) the modulus of elasticity (lb/in^2) - E
5) the yield stress (lb/in^2) - SIGYP

Selecting the number of elements, NI, we number the displacements and slopes at element boundaries. This must be done as indicated in Figure 4.

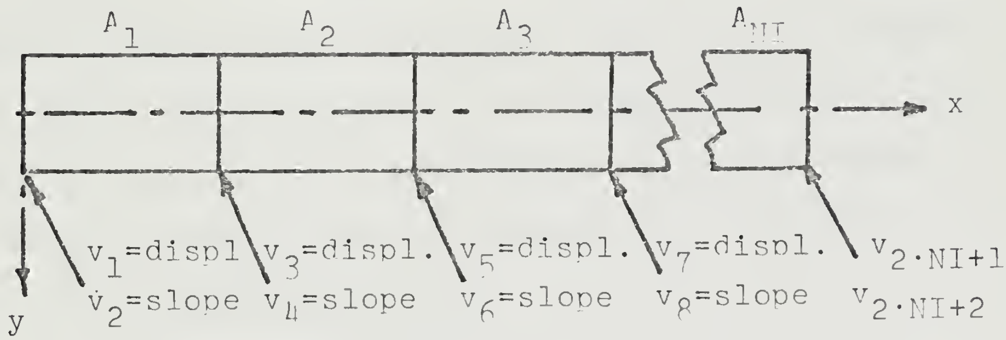


Figure 4. The Finite Element Numbering Scheme.

We consider the boundary conditions of the problem and form two vectors. One, called IBC, is a vector of subscripts of the components of the displacement/slope vector, V, which are fixed. These are ordered in increasing value. The second, called BC, is a vector of the values of these fixed components. The number of boundary conditions is called NBC. For example, if the displacements at the left and right ends of the beam are fixed at zero and the slopes at these points are variable (simply supported), the vectors are:

IBC(1) = 1	BC(1) = 0
IBC(2) = (2 x NI) + 1	BC(2) = 0

The total number of parameters is equal to (3 x NI) + 3 and is denoted NT. Subtracting the number of boundary conditions, NBC, we obtain the total number of unknowns, NU = NT - NBC. This is also the size of the reduced system.

From the given loading condition on the beam we form the global consistent load distribution vector $<\bar{D}>$ for the number of elements chosen. This is denoted CLOAD in the program.

The initial guess vector, X, is formed by guessing the magnitude of the unknown displacements and slopes, the areas of the elements, and the maximum load intensity. These must be ordered as follows:

1) Displacements and slopes as ordered in the vector $\{\bar{v}\}$ with boundary conditions removed.
2) Element areas starting at the origin.
3) Maximum load intensity.

We select values for MAXIT, NUMSIG, and IPRINT from their descriptions in SUBROUTINE NLNSYS.

A calling program is written which dimensions X (the starting guess vector). This dimension is NU. The program calls SUBROUTINE OPBEAM with numerical values for the arguments except for X. A sample calling program is:

```
                *
      DIMENSION X(NU)
                  *     *     *     *
      CALL OPBEAM(NU,MAXIT,NUMSIG,IPRINT,X)
      STOP
      END
```

*(numerical values)

The user must provide the two statements for SUBROUTINE FCNLST as described in the program. The first is the dimension statement:

```
      DIMENSION IBC(NBC),BC(NBC),CLOAD(2xNI+2),G(NT),
      H(NT),V(2xNI+2),A(NI)
```

The dimensions are numerical values.

The second is the data statement:

```
      DATA NU/*/,NBC/*/,NI/*/,NT/*/
```

where * indicates the numerical values for the parameters.

Data cards are prepared and inserted in the following order and format:

1) The guess vector X - 6E12.5
2) The physical constants:
   BLENG,E,HI,VOL,SIGYP - 5E12.5
3) The boundary conditions:

IBC(1), BC(1) - I10,E12.5

IBC(2), BC(2) - I10,E12.5

⋮

IBC(NBC),BC(NBC) - I10,E12.5

4) The consistent load distribution vector

CLOAD - 6E12.5

The final output may be one of the following:

1) A singularity was generated at the output vector.
2) The iteration limit was reached.
3) A solution was reached.

In case 1 a new starting guess should be made and the problem re-run. In case 2, if the number of iterations was small (<30), try a larger number of iterations. If it was not small, try a new starting guess. Many guesses may be required before a solution is reached.

An example problem set-up is shown in Appendix D along with the results obtained for different problems.

V. RESULTS, CONCLUSIONS, AND RECOMMENDATIONS

A. RESULTS

Figures 22, 26, and 28 compare the three element finite element optimum shape with the exact optimum shape for a specific beam under a specific loading condition. Table I compares the maximum load intensities for the finite element designs of Figures 22-28 with those for the beam of equal volume, length, and height but constant cross-section.

TABLE I

COMPARISON OF MAXIMUM LOAD RESULTS

Loading	A	B	C	D	E
Simply Supported Uniform Load	510 lb/in	764 lb/in	721 lb/in	141	5.6
Concentrated Loads at					
x = 10 in	100 KIP	-	194 KIP	194	-
x = 20 in	55 KIP	-	105 KIP	191	-
x = 30 in	40.7 KIP	-	76 KIP	187	-
x = 40 in	34.4 KIP	-	63 KIP	183	-
x = 50 in	31.4 KIP	-	58 KIP	181	-
x = 60 in	30.6 KIP	-	57 KIP	186	6.6
Triangular Load	996 lb/in	-	1839 KIP	185	-
Cantilever Beam Uniform Load	191 lb/in	382 lb/in	360 lb/in	188	5.6

A = Maximum Load for Constant Area Beam
B = Maximum Load for Exact Optimum Beam
C = Maximum Load for Three Element Beam
D = % Increase of Load from Case A to Case C
E = % Deviation of Case C from Case B

It is clearly evident in Figures 22, 26, and 18 that the three-element design approximates the exact optimum shape. The best criterion for the accuracy of the approximation, however, is not the beam's appearance. Since our goal is to maximize the load intensity, the accuracy of the approximation is best measured by the closeness of the load intensity allowed by the finite element design to that allowed by the exact optimum design.

In those examples where exact solutions were known, even the coarse three-element approximation allows a maximum load intensity that is close to that of the exact optimum shape. Table I shows that the three-element design gives maximum loads within 5.6% of the exact design maximum for the simply supported and cantilever beams under uniform load and within 6.6% for the simply supported beam under a concentrated load at its center.

Attempts were made to increase the number of elements in the simply supported beam under a uniform load. This was desired in order to show that the shape would converge to the exact solution with an increasing number of elements. However, difficulty was encountered in finding an initial guess vector within the range of convergence. Due to the limited time available and the need to check the model's validity for other loading and boundary conditions, the three-element model was used throughout.

B. CONCLUSIONS

The above results show that the finite element model developed in Section III, for the structural optimization of a homogeneous beam of uniform fixed height, and fixed volume, is valid. Furthermore, the results show that a relatively small number of elements may be used to design a beam that will closely approximate the strength of an exact maximum strength design.

As a practical matter, the construction of a beam with the shape of the finite element solution is much easier than constructing the exact optimum shape. In the exact optimum design, points where the cross-sectional area is zero must have some material added to account for the shear effects that were neglected in the formulation. Much less, if any, material must be added to the finite element design at the same points. The finite element design would have stress concentrations at the element boundaries which are not present in the exact optimum design.

The finite element model provides a means of approximating solutions to optimization problems which are impossible to solve in a closed form by classical techniques and extremely difficult to solve numerically. This is due mainly to the algebraic nature of the finite element method and the differential equation nature of the classical methods.

The model developed was, of necessity, for the simplest type of beam and loading conditions. The approach used,

however, should be valid, with some modifications, for the more complex cross-sections. It has been shown that the finite element techniques can be successfully applied to the problems of beam optimization.

C. RECOMMENDATIONS

Many more problems for which the developed model applies should be solved with various combinations of loading and boundary conditions for a parametric study of the beam optimization problem. Problems with more elements should be solved to show convergence to known exact solutions.

Although SUBROUTINE NLNSYS provided solutions, the algorithm is very sensitive to the starting guess in relation to the region of convergence. More than one attempt is usually required to obtain a solution to a given problem. An algorithm that does not exhibit this sensitivity is required to enable more efficient problem solving using finite element techniques. At the present time very few algorithms for solving systems of nonlinear equations exist.

Another technique that is possible would be to use some type of direct search or steepest descent algorithm to find an optimum solution to the finite element potential energy functional itself. With this type of formulation it may be possible to allow variable element areas. A solution might be more readily attained as compared with a solution to the system of nonlinear equations. Some work was started in this direction late in the investigation. The initial

results looked promising but time restrictions dictated that this approach be terminated in favor of completing the presented formulation.

Models for other cross section types could be developed using a similar approach to that presented in Section III. Also models with other than equal size elements or for beams of more than one material would be of interest.

In the laboratory, an experiment could be run comparing the behavior of an exact optimum beam with that of its finite element approximations to see if the results compare with those predicted by the models.

The entire area of the application of finite element methods to optimization has hardly been touched. The finite element method is a powerful tool and more research should be done in its application in this field.

APPENDIX A

EXAMPLES OF BEAM STRUCTURAL OPTIMIZATION PROBLEMS USING THE VARIATIONAL FORMULATION

1. Consider a simply supported beam of rectangular cross-section with constant height, h, and variable width, b(x). The beam is under a uniform load distribution and has given volume and length, V_o and L respectively. Find the optimum shape, b(x) such that the load intensity P_o is a maximum. The beam has a modulus of elasticity E and yield strength S_y.

Solution:

The moment of inertia of the cross-section is:

$$I = \frac{bh^3}{12} = \frac{h^2}{12}(bh) = \frac{h^2}{12} \cdot A$$

Thus $C = h^2/12$ and $n = 1$. Using equation (II-8) with $n = 1$ obtain:

$$A = \frac{V_o(Lx-x^2)}{\int_0^L (Lx-x^2)dx}$$

Invoking equation (II-10) to obtain (II-11) the value of λ becomes:

$$\lambda = \frac{S_y^2}{6E}$$

Substituting the values for C, λ, and n into equation (II-9) yields:

$$P_o = \frac{hS_yV_o}{3\int_0^L (Lx-x^2)dx}$$

The integral can be evaluated in closed form.

$$\int_0^L (Lx-x^2)\,dx = L^3/6$$

Thus:

$$A(x)_{opt} = \frac{6V_o}{L^3}(Lx-x^2)$$

which gives:

$$b(x)_{opt} = \frac{6V_o}{hL^3}(Lx-x^2)$$

The beam shape is graphically depicted in Figure 5. The maximum load intensity is:

$$P_{o_{max}} = \frac{2hS_yV_o}{L^3}$$

This shows that, for the given problem, maximum load intensity varies linearly with the height, yield stress, and volume, and inversely with the cube of length.

2. Consider a simply supported beam of rectangular cross-section with a constant width, b, and variable height, h(x). The beam is under a uniform load intensity P_o and has a given volume, V_o, and length, L. Find the optimum shape, h(x) such that the load intensity, P_o, is a maximum. The beam has a modulus of elasticity, E, and yield strength, S_y.

Solution:

The moment of inertia of the cross-section is:

$$I = \frac{bh^3}{12} = \frac{1}{12b^2} A^3$$

Thus $C = \frac{1}{12b^2}$ and $n = 3$. Using equation (II-8) with $n = 3$:

$$A = \frac{V_o (Lx-x^2)^{\frac{1}{2}}}{\int_0^L (Lx-x^2)^{\frac{1}{2}} dx}$$

Equation (II-13) gives :

$$\lambda = S_y^2/2E$$

With λ, C, and n above, equation (II-9) becomes:

$$P_o = \frac{S_y V_o^2}{3b} \cdot \frac{1}{\left[\int_0^L (Lx-x^2)^{\frac{1}{2}} dx\right]^2}$$

The integral may be evaluated and is:

$$\int_0^L (Lx-x^2)^{\frac{1}{2}} dx = \frac{\pi L^2}{8}$$

Thus:

$$A_{opt} = \frac{8V_o (Lx-x^2)^{\frac{1}{2}}}{\pi L^2} \simeq \frac{2.55\ V_o (Lx-x^2)^{\frac{1}{2}}}{L^2}$$

Solving for h(x):

$$h(x) \simeq \frac{2.55\ V_o (Lx-x^2)^{\frac{1}{2}}}{bL^2}$$

h(x) is graphically depicted in Figure 6. The maximum permissible load intensity is:

$$Po_{max} = \frac{64 S_y V_o^2}{s^2 b L^4} \approx \frac{2.17 S_y V_o^2}{b L^4}$$

Here Po_{max} varies linearly with yield stress, parabolically with volume and inversely with width and the fourth power of length.

3. Consider a simply supported beam of circular cross-section under a uniform load distribution. It has a given volume and length, V_o and L respectively. Find the optimum radius, r(x) such that the load intensity is a maximum. The beam has a modulus of elasticity, E, and yield strength, S_y.

<u>Solution</u>:

The moment of inertia of a circular cross-section is:

$$I = \frac{\pi r^4}{4} = \frac{1}{4\pi} (\pi r^2)^2 = CA^n$$

Thus $C = \frac{1}{4\pi}$ and n = 2. With n = 2, equation (II-8) becomes

$$A = \frac{V_o (Lx-x^2)^{2/3}}{\int_0^L (Lx-x^2)^{2/3} dx}$$

Equation (II-12) gives the value of λ for a circular cross-section as:

$$\lambda = \frac{S_y^2}{4E}$$

With λ, C, and n as above, equation (3-9) becomes:

$$P_o = \frac{S_y V_o^{2/3}}{2\sqrt{\pi}} \cdot \frac{1}{\left[\int_0^L (Lx-x^2)^{2/3} dx\right]^{3/2}}$$

The integral may be evaluated in closed form. However it involves the Gamma Function which must be approximated. The integral is, then:

$$\int_0^L (Lx-x^2)^{2/3} dx \approx .0979 \cdot (L)^{7/3}$$

Thus:

$$A_{opt} \approx \frac{V_o(Lx-x^2)^{2/3}}{.0979L^{7/3}}$$

Solving $A = \pi r^2$ for r:

$$r(x)_{opt} \approx \sqrt{\frac{V_o(Lx-x^2)^{2/3}}{.308\ L^{7/3}}}$$

The beam shape is depicted in Figure 7. The maximum load intensity is given by:

$$Po_{max} \quad \frac{S_y V_o^{3/2}}{.1067\ L^{7/2}}$$

Po_{max} is linear in yield strength, proportional to the 7/2 power of volume and inversely proportional to the 7/2 power of length.

Reference [11] gives optimum beam shapes for example problems 1 and 2 derived in a different manner. The shapes

shown in Figures 5 and 6 are identical to those shapes. The maximum load intensities also correspond.

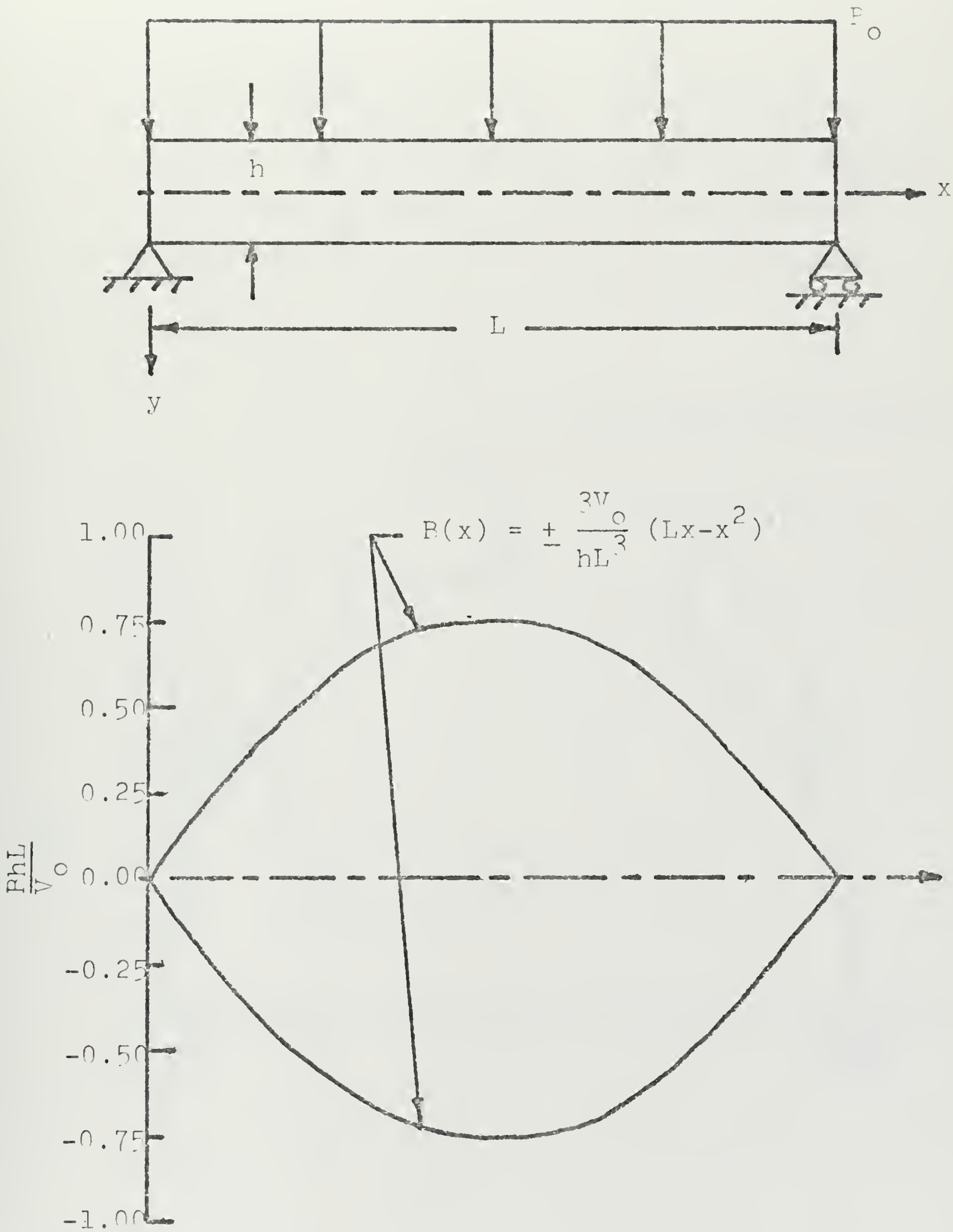


Figure 5. Optimum Beam Shape for a Rectangular Cross-section Beam of Uniform Height under a Uniform Load.

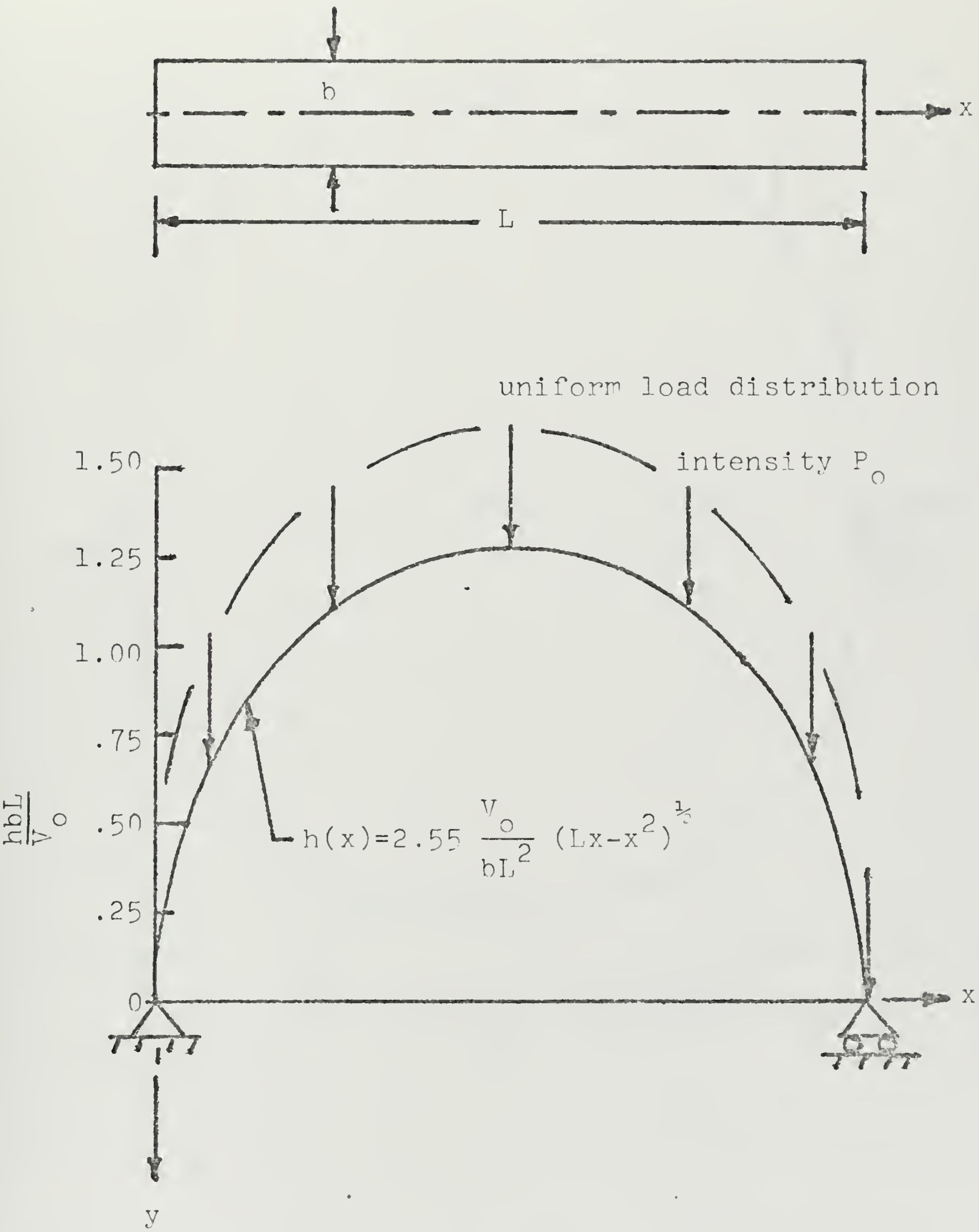


Figure 6. Optimum Beam Shape for a Rectangular Cross-section Beam of Uniform Width under Uniform Load.

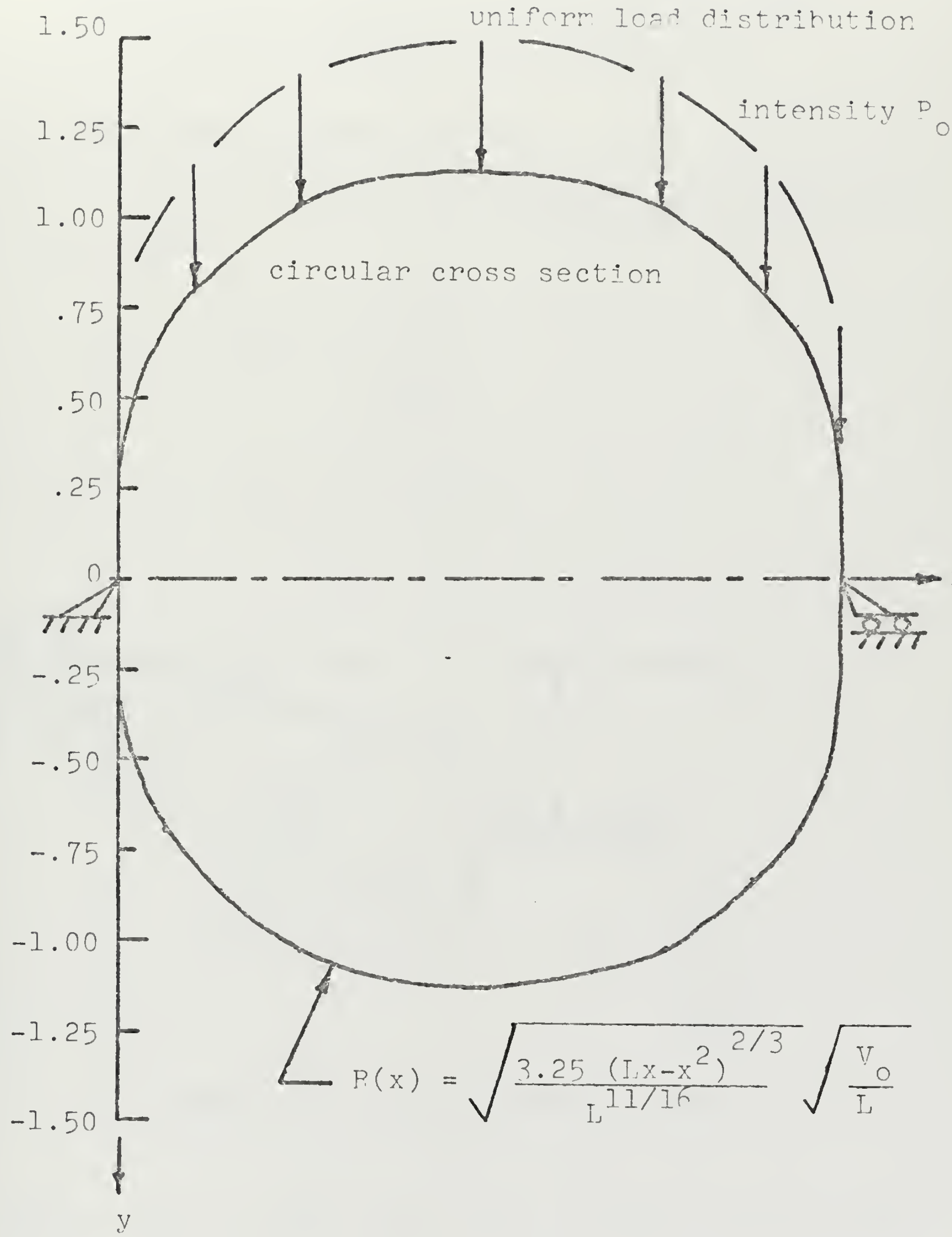


Figure 7. Optimum Beam Shape for a Circular Cross-section Beam under a Uniform Load.

APPENDIX B

CALCULATIONS FOR THE FINITE ELEMENT METHOD FOR BEAM STRUCTURAL OPTIMIZATION

1. The Cubic Displacement Function

We wish to find a cubic displacement function vector $\langle N(x_i)\rangle$ such that

$$v(x_i) = \langle N(x_i)\rangle \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

where $v(x_i)$ is a cubic polynomial.

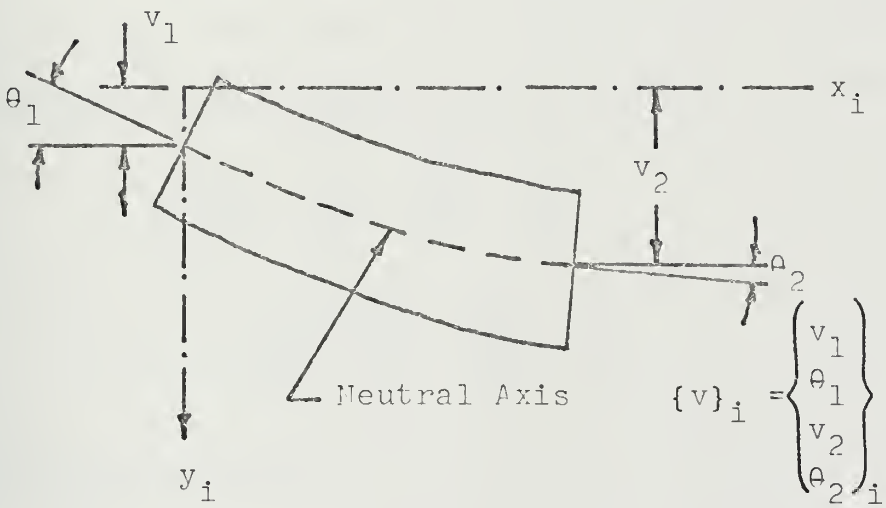


Figure 8. The Deflected Element.

We let $v(x_i) = \alpha x_i^3 + \beta x_i^2 + \gamma x_i + \delta$ over the interval $0 \leq x_i \leq \Delta$. Boundary conditions are:

(1) $v(0) = v_1$

(2) $v'(0) = \theta_1$

(3) $v(\Delta) = v_2$

(4) $v'(\Delta) = \theta_2$

Substituting boundary condition (1) in $v(x_i)$ gives $\delta = v_1$.

Taking the first derivative of $v(x_i)$ with respect to x_i yields:

$$v'(x_i) = 3\alpha x_i^2 + 2\beta x_i + \gamma$$

Boundary condition (2) in $v'(x_i)$ yields:

$$\gamma = \theta_1$$

Substituting the values for γ and δ into v, and v' and invoking boundary conditions (3) and (4) yields:

$$\alpha = \frac{2v_1}{\Delta^3} + \frac{\theta_1}{\Delta^2} - \frac{2v_2}{\Delta^3} + \frac{\theta_2}{\Delta^2}$$

$$\beta = -\frac{3v_1}{\Delta^2} - \frac{2\theta_1}{\Delta} + \frac{3v_2}{\Delta^2} - \frac{\theta_2}{\Delta}$$

In vector notation:

$$\alpha = \left\langle \frac{2}{\Delta^3} \; \frac{1}{\Delta^2} \; - \frac{2}{\Delta^3} \; \frac{1}{\Delta^2} \right\rangle \{v\}_i$$

$$\beta = \left\langle - \frac{3}{\Delta^2} \; - \frac{2}{\Delta} \; \frac{3}{\Delta^2} \; - \frac{1}{\Delta} \right\rangle \{v\}_i$$

$$\gamma = \langle 0 \; 1 \; 0 \; 0 \rangle \{v\}_i$$

$$\delta = \langle 1 \; 0 \; 0 \; 0 \rangle \{v\}_i$$

Thus in vector notation:

$$v(x_i) = \left\langle \left(\frac{2x_i^3}{\Delta^3} - \frac{3x_i^2}{\Delta^2} + 1\right)\left(\frac{x_i^3}{\Delta^2} - \frac{2x_i^2}{\Delta} + x_i\right)\left(-\frac{2x_i^3}{\Delta^3} + \frac{3x_i^2}{\Delta^2}\right)\left(\frac{x_i^3}{\Delta^2} - \frac{x_i^2}{\Delta}\right)\right\rangle \{v\}_i$$

Therefore the shape function vector $<N(x_i)>$ is:

$$\underset{1x4}{<N(x_i)>} = \left\langle \left(\frac{2x_i^3}{\Delta^3} - \frac{3x_i^2}{\Delta^2} + 1\right)\left(\frac{x_i^3}{\Delta^2} - \frac{2x_i^2}{\Delta} + x_i\right)\left(-\frac{2x_i^3}{\Delta^3} + \frac{3x_i^2}{\Delta^2}\right)\left(\frac{x_i^3}{\Delta^2} - \frac{x_i^2}{\Delta}\right)\right\rangle$$

The second derivative of this vector with respect to x_i is:

$$<N(x_i)''> = \left\langle \left(\frac{12x_i}{\Delta^3} - \frac{6}{\Delta^2}\right)\left(\frac{6x_i}{\Delta^2} - \frac{4}{\Delta}\right)\left(-\frac{12x_i}{\Delta^3} + \frac{6}{\Delta^2}\right)\left(\frac{6x_i}{\Delta^2} - \frac{2}{\Delta}\right)\right\rangle$$

2. <u>The Modified Element Stiffness Matrix</u>

The modified element stiffness matrix is defined as:

$$\underset{4x4}{[k^*]} = \int_0^\Delta \underset{4x1}{<N''(x_i)>^T} \; \underset{1x4}{<N''(x_i)>} \, dx_i$$

Taking the indicated vector product and integrating yields:

$$[k^*] = \begin{bmatrix} 12/\Delta^3 & 6/\Delta^2 & -12/\Delta^3 & 6/\Delta^2 \\ 6/\Delta^2 & 4/\Delta & -6/\Delta^2 & 2/\Delta \\ -12/\Delta^3 & -6/\Delta^2 & 12/\Delta^3 & -6/\Delta^2 \\ 6/\Delta^2 & 2/\Delta & -6/\Delta^2 & 4/\Delta \end{bmatrix}$$

3. <u>The Consistent Load Distribution Vector</u>

Assume that the load per unit length over the i^{th} element may be written as an r^{th} order polynomial

$$P(x_i) = \bar{\alpha} + \bar{\beta}x_i + \bar{\gamma}x_i^2 + \ldots + \bar{\eta}x^r$$

where $\bar{\alpha}$, $\bar{\beta}$, $\bar{\gamma}$,...,$\bar{\eta}$ are constants which are determined from known values of load intensity at r+1 loactions in $0 \leq x_i \leq \Delta$. These known values are represented as fractions of the maximum load intensity, P_o, over the beam $(0 \leq x \leq L)$ such as $\bar{\alpha} = \alpha \cdot \frac{P(x_i)}{P_o}$ where $0 \leq \bar{x}_i \leq \Delta$. Thus:

$$P(x_i) = P_o[\alpha + \beta x_i + \gamma x_i^2 + \ldots + \eta x^r]$$

The load on an increment dx_i of the i^{th} element is:

$$dP(x_i) = P_o[\alpha + \beta x_i + \gamma x_i^2 + \ldots + \eta x_i^r]\ dx_i$$

The work done on the i^{th} element by this incremental force is:

$$dw_i = dP(x_i)\ v(x_i)$$

but:

$$v(x_i) = \underset{1x4}{<N(x_i)>}\ \underset{4x1}{\{v\}_i}$$

Thus:

$$dw_i = P_o[\alpha+\beta x_i+\gamma x_i^2+\ldots+\eta x_i^r]\ <N(x_i)>\ \{v\}_i\ dx_i$$

The work done on the i^{th} element is:

$$w_i = \int_0^\Delta dw_i$$

$$w_i = \int_0^\Delta P_o[\alpha+\beta x_i+\gamma x_i^2+\ldots+\eta x_i^r]\ <N(x_i)>\ \{v\}_i\ dx_i$$

Letting $[\alpha+\beta x_i+\gamma x_i^2+\ldots+\eta x_i^r]$ be denoted by $Q_r(x_i)$ and noting that P_o and $\{v\}_i$ are independent of x_i, the above expression becomes:

$$w_i = P_o \int_0^{\Delta} Q_r(x_i) <N(x_i)> dx_i \{v\}_i$$

The integral is defined as the element consistent load distribution vector $<D>_i$

$$\underset{1x4}{<D>_i} = \int_0^{\Delta} Q_r(x_i) \underset{1x4}{<N(x_i)>} dx_i$$

The work done on the i^{th} element may then be written:

$$w_i = P_o <D>_i \{v\}_i$$

The work done on the beam is

$$W = \sum_{i=1}^{n} w_i = P_o \sum_{i=1}^{n} <D>_i \{v\}_i$$

It may be noted there is no requirement for the element load distribution polynomial, $P(x_i)$, to be the same for each element. On the contrary, each element may have a different load distribution polynomial, $P(x_i)$, and the distribution may even be discontinuous at element boundaries. The only requirement is that the ratio of $P(\text{local})/P_o$ be known at r+1 points over the element in order to define the r^{th} order distribution polynomial.

We take the following case as an example:

Consider the first three elements of an n element beam. Given that the distributions of load on elements 1,

2, and 3 are cubic, linear and quadratic respectively, and also given the values of P_z/P_o where z = A, B, C, ...,H, we seek an expression for the work done on the first three elements of the beam. Figure 9 shows a graphical representation of the problem.

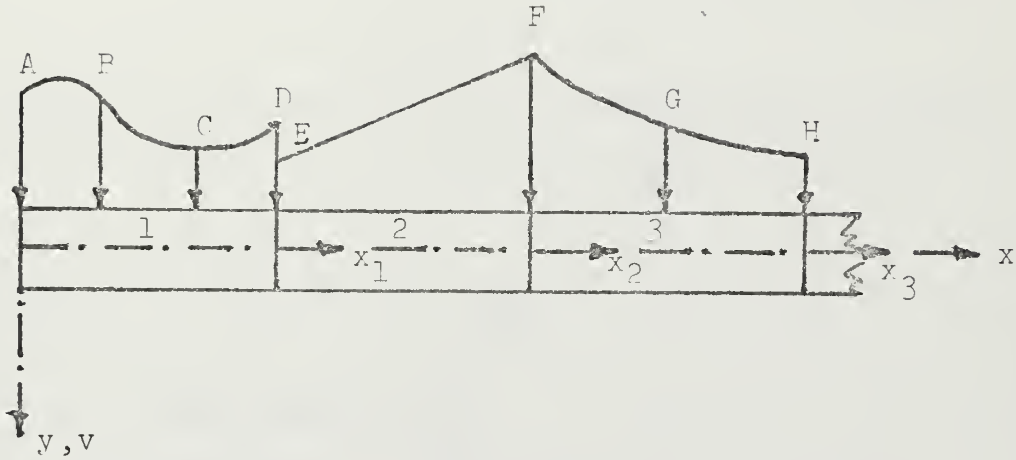


Figure 9. Multiple Loading Types on a Beam.

The values P_A/P_o through P_D/P_o define the cubic distribution function $Q_3(x_1)$ over the first element. The values P_E/P_o and P_F/P_o define the linear distribution function $Q_1(x_2)$ over the second element. The values of P_F/P_o through P_H/P_o define the quadratic distribution function $Q_2(x_3)$ over the third element. These functions in turn define consistent load distribution vectors $\langle D_1 \rangle$, $\langle D_2 \rangle$, and $\langle D_3 \rangle$ for the elements.

$$\langle D_1 \rangle = \int_0^{\Delta} Q_3(x_1) \langle N(x_1) \rangle \, dx_1$$

$$\langle D_2 \rangle = \int_0^{\Delta} Q_1(x_2) \langle N(x_2) \rangle \, dx_2$$

$$\langle D_3 \rangle = \int_0^{\Delta} Q_2(x_3) \langle N(x_3) \rangle \, dx_3$$

The work done on the first three elements is:

$$W(1\text{-}3) = P_o[\langle D\rangle_1 \{\bar{v}_1\} + \langle D\rangle_2 \{\bar{v}_2\} + \langle D\rangle_3 \{\bar{v}_3\}]$$

4. The Concentrated Load

If the load is concentrated vice distributed, the consistent load distribution vector is formed in a manner parallel to that of the distributed load.

We apply a concentrated load of intensity $\beta P_o (0 < \beta \le 1)$ at a distance $\alpha\Delta (0 \le \alpha \le 1)$ from the origin of the i^{th} element.

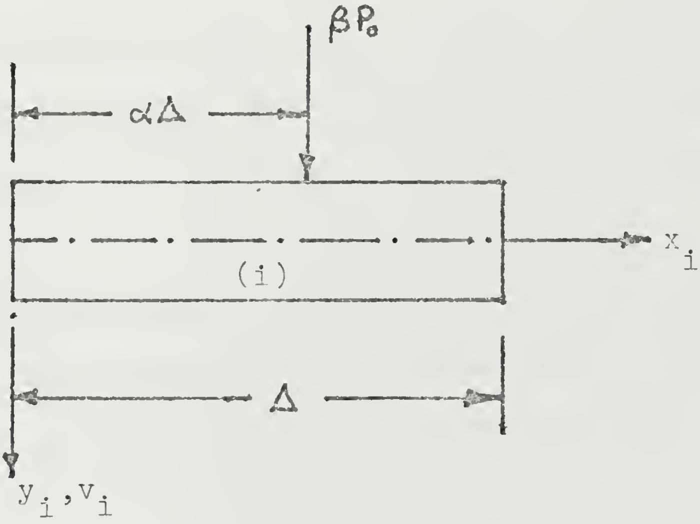


Figure 10. A General Element with Concentrated Load.

The work done on this element is:

$$w_i = \beta P_o v(\alpha\Delta)$$

But:

$$v(\alpha\Delta) = \langle N(\alpha\Delta)\rangle \{v\}_i$$

Therefore:

$$w_i = \beta P_o \; \langle N(\alpha\Delta)\rangle \; \{v\}_i$$

If we define the element consistent load distribution vector as:

$$\langle D\rangle_i = \beta \; \langle N(\alpha\Delta)\rangle$$

we obtain for the work done on the i^{th} element by the load P_o:

$$w_i = P_o \; \langle D\rangle_i \; \{v\}_i$$

Note that the expression is the same as that for the distributed load. However, the units for P_o and $\langle D\rangle_i$ are different - P_o is in units of force vice force per unit length.

If multiple concentrated loads are applied to one element, the element consistent load vector is the sum of those obtained by considering each load separately. In symbolic form:

The individual load vectors are:

$$\langle D_1\rangle_i = \beta_1 \; \langle N(\alpha_1\Delta)\rangle$$

$$\langle D_2\rangle_i = \beta_2 \; \langle N(\alpha_2\Delta)\rangle$$

$$\vdots$$

$$\langle D_j\rangle_i = \beta_j \; \langle N(\alpha_j\Delta)\rangle$$

The element consistent load vector is:

$$\langle D\rangle_i = \sum_j \beta_j \; \langle N(\alpha_j\Delta)\rangle$$

5. The Finite Element Variational Equations.

We have as the global augmented potential energy functional:

$$T_g^* = \frac{EC}{2} \sum_{i=1}^{n} A_i \{\bar{v}_i\}^T [k^*] \{\bar{v}_i\} - P_o \sum_{i=1}^{n} \langle D\rangle_i \{\bar{v}_i\}$$

$$- \lambda \sum_{i=1}^{n} \Delta A_i$$

where n is the number of elements.

Define the following as the global consistent load distribution vector $\underset{1x(2n+2)}{\langle\bar{D}\rangle}$

$$\langle\bar{D}\rangle = \langle D_1^1, D_1^2, D_1^3 + D_2^1, D_1^4 + D_2^2, \ldots, D_{n-1}^4 + D_n^2, D_n^3, D_n^4\rangle$$

$$= \langle\bar{D}_1, \bar{D}_2, \bar{D}_3, \ldots, \bar{D}_{2n+1}, \bar{D}_{2n+2}\rangle$$

$\langle\bar{D}\rangle$ is formed by assembling the element distribution vectors as indicated. Thus the global work term becomes:

$$P_o \sum_{i=1}^{n} \langle D_i\rangle \{\bar{v}_i\} = P_o \underset{1x(2n+2)}{\langle\bar{D}\rangle} \underset{(2n+2)x1}{\{\bar{v}\}}$$

The augmented global potential energy functional that will be operated on is:

$$T_g^* = \frac{EC}{2} \sum_{i=1}^{n} A_i \{\bar{v}_i\}^T [k^*] \{\bar{v}_i\} - P_o \langle\bar{D}\rangle \{\bar{v}\} - \lambda\Delta \sum_{i=1}^{n} A_i$$

Taking the partial derivative of T* with respect to each component v_j in $\{\bar{v}\}$ and setting result equal to zero gives:

$$\frac{\partial T_g^*}{\partial v_1} = 0: \quad ECA_1 \sum_{j=1}^{4} k_{1j}^* \bar{v}_j - P_o\bar{D}_1 = 0$$

$$\frac{\partial T_g^*}{\partial v_2} = 0: \quad ECA_1 \sum_{j=1}^{4} k_{2j}^* \bar{v}_j - P_o \bar{D}_2 = 0$$

$$\frac{\partial T_g^*}{\partial v_{(i-1)}} = 0: \quad ECA_{(\frac{i-2}{2})} \sum_{j=1}^{4} k_{3j}^* v_{(j+i-4)}$$

$$+ ECA_{(i/2)} \sum_{j=1}^{4} k_{1j}^* v_{(j+i-2)} - P_o \bar{D}_{(i-1)} = 0$$

$$i = 4,6,8,\ldots 2n$$

$$\frac{\partial T_g^*}{\partial v_i} = 0: \quad ECA_{(\frac{i-2}{2})} \sum_{j=1}^{4} k_{4j}^* v_{(j+i-4)}$$

$$+ ECA_{(i/2)} \sum_{j=1}^{4} k_{2j}^* v_{(j+i-2)} - P_o \bar{D}_i$$

$$i = 4,6,8,\ldots 2n$$

$$\frac{\partial T_g^*}{\partial v_{(2n+1)}} = 0: \quad ECA_n \sum_{j=1}^{4} k_{3j}^* v_{(j+2n-2)} - P_o \bar{D}_{(2n+1)} = 0$$

$$\frac{\partial T_g^*}{\partial v_{(2n+2)}} = 0: \quad ECA_n \sum_{j=1}^{4} k_{4j}^* v_{(j+2n-2)} - P_o \bar{D}_{(2n+2)} = 0$$

This gives 2n+2 equations.

We next take the partial derivatives of T_g^* with respect to the A_i's and set them equal to zero:

$$\frac{\partial T_g}{\partial A_i} = 0: \quad \sum_{j=1}^{4} \sum_{k=1}^{4} k_{ij}^* v_{(k+2i-2)} v_{(j+2i-2)} - \frac{2\Delta}{EC}$$

$$i = 1,2,\ldots,n$$

This gives n equations.

At this stage we have 2n+2 equations in 3n+3 unknowns hence an additional equation is necessary. This equation is the volume constraint:

$$\Delta \sum_{i=1}^{n} A_i - V_o = 0$$

Therefore we have a total of 3n+3 equations in the 3n+3 unknowns:

2n+2 components of the vector $\{\bar{v}\}$

n areas A_i

1 maximum load intensity P_o

We therefore have a mathematically tractable problem.

APPENDIX C

EXAMPLES OF CONSISTENT LOAD DISTRIBUTION VECTORS

1. As our first example we consider a three element problem under a uniform load distribution.

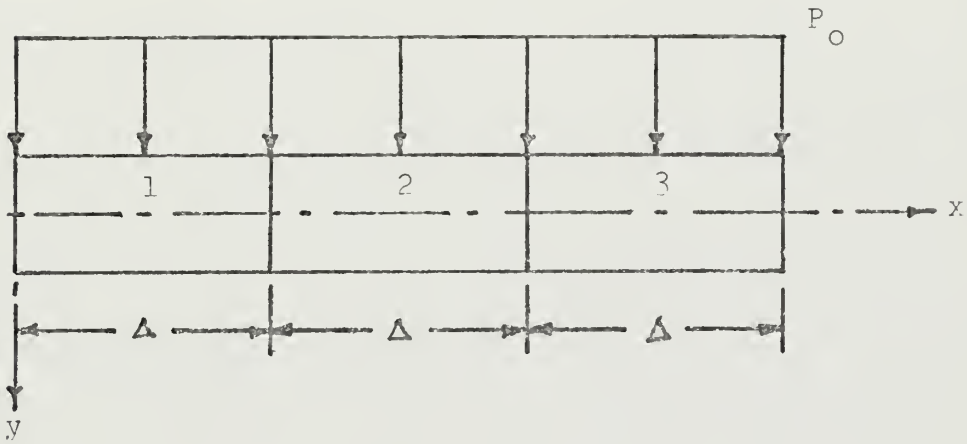


Figure 11, A Three Element Beam, Uniform Load.

Observing the general element of Figure 12, the work done on this element is:

$$w_i = \int_0^{\Delta} P_o <N(x_i)> dx_i \{v\}_i$$

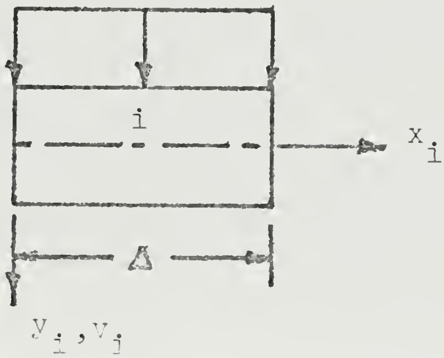


Figure 12. A General Element, Uniform Load.

which may be written:

$$w_i = P_o \int_0^{\Delta} <N(x_i)> dx_i \; \{v\}_i$$

From our previous definition in Appendix B, the element consistent load distribution vector is:

$$<D>_i = \int_0^{\Delta} <N(x_i)> dx_i$$

Evaluating the integral where $<N(x_i)>$ is as defined in Appendix B:

$$\int_0^{\Delta} <N(x_i)> dx_i = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, - \frac{\Delta^2}{12} >$$

Thus

$$<D>_i = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, - \frac{\Delta^2}{12} > \qquad \text{(C-1)}$$

Applying this to the three element problem and noting that the load intensity is the same for each element, we obtain:

$$<D>_1 = <D>_2 = <D>_3 = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, - \frac{\Delta^2}{12} >$$

We assemble these element vectors to obtain the global consistent load distribution vector $<\bar{D}>$

$$<\bar{D}> = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \Delta, 0, \Delta, 0, \frac{\Delta}{2}, - \frac{\Delta^2}{12} > \qquad \text{(C-2)}$$

The work done on the beam is:

$$W = P_o < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \Delta, 0, \Delta, 0, \frac{\Delta}{2}, - \frac{\Delta^2}{12} > \begin{Bmatrix} \bar{v}_1 \\ \bar{v}_2 \\ \bar{v}_3 \\ \bar{v}_4 \\ \bar{v}_5 \\ \bar{v}_6 \\ \bar{v}_7 \\ \bar{v}_8 \end{Bmatrix}$$

2. As a second example let us take the following problem:

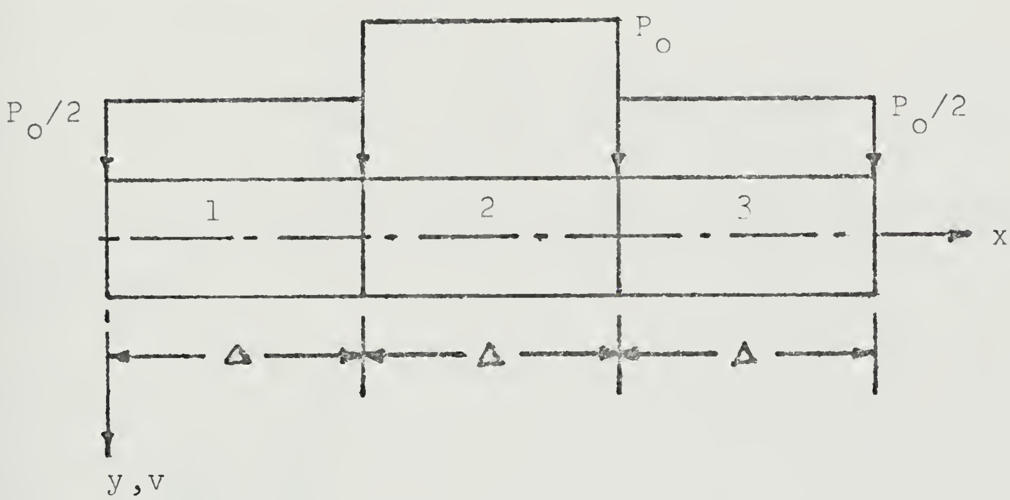


Figure 13. Three Element Beam, Multiple Uniform Loads.

The element consistent load distribution vector for a uniform load from equation (C-1) is:

$$<D>_i = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, - \frac{\Delta^2}{12} >$$

In this problem, however, the load intensity is not the same on each element. Therefore the element consistent load distribution vectors are:

$$\langle D \rangle_1 = \langle D \rangle_3 = \frac{1}{2} \langle D \rangle_i$$

$$\langle D \rangle_2 = \langle D \rangle_i$$

Or:

$$\langle D \rangle_1 = \langle D \rangle_3 = \langle \frac{\Delta}{4}, \frac{\Delta^2}{24}, \frac{\Delta}{4}, -\frac{\Delta^2}{24} \rangle$$

$$\langle D \rangle_2 = \langle \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, -\frac{\Delta^2}{12} \rangle$$

Assembling these vectors we obtain for the global consistent load distribution vector:

$$\langle \bar{D} \rangle = \langle \frac{\Delta}{4}, \frac{\Delta^2}{24}, \frac{3\Delta}{4}, \frac{\Delta^2}{12}, \frac{3\Delta}{4}, \frac{-\Delta^2}{12}, \frac{\Delta}{4}, \frac{-\Delta^2}{24} \rangle$$

3. Let us now look at the following problem with a concentrated load:

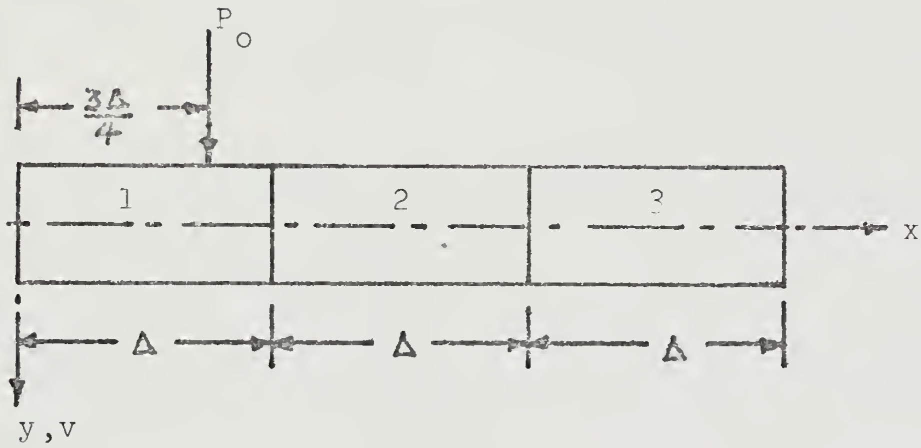


Figure 14. Three Element Beam, One Concentrated Load.

Taking a general element with a concentrated load:

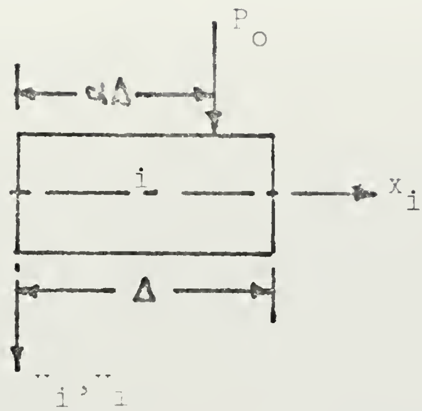


Figure 15. General Element, Concentrated Load.

The consistent load distribution vector for this element is:

$$\langle D\rangle_i = \langle N(\alpha\Delta)\rangle$$

But:

$$\langle N(\alpha\Delta)\rangle = \langle (2\alpha^3-3\alpha^2+1),\ (\alpha^3\Delta-2\alpha^2\alpha+\alpha\Delta),\ (-2\alpha^3+3\alpha^2),\ (\alpha^3\Delta-\alpha^2\Delta)\ \rangle$$

Thus:

$$\langle D\rangle_i = \langle (2\alpha^3-3\alpha^2+1),\ [(\alpha\Delta)(\alpha^2-2\alpha+1)],\ [(\alpha^2)(-2\alpha+3)],\ [(\alpha^2\Delta)(\alpha-1)]\ \rangle \qquad (C-4)$$

In the specified problem there are no loads on elements 2 or 3 thus:

$$\langle D\rangle_2 = \langle D_3\rangle = \langle 0,\ 0,\ 0,\ 0\rangle$$

The element consistent load vector for element 1 is therefore:

$$\langle D\rangle_1 = \langle D(\tfrac{3\Delta}{4})\rangle_i = \langle\ \tfrac{5}{32},\ \tfrac{3\Delta}{64},\ \tfrac{27}{32},-\ \tfrac{9\Delta}{64}\ \rangle \qquad (C-4a)$$

Assembling $<D>_1$, $<D>_2$, and $<D_3>$ we obtain the global consistent load distribution vector $<\bar{D}>$.

$$<\bar{D}> = < \frac{5}{32}, \frac{3\Delta}{64}, \frac{27}{32}, -\frac{9\Delta}{64}, 0, 0, 0, 0 > \qquad \text{(C-5)}$$

4. Let us take example 3 and add an additional load on element 1 of intensity P_o a distance $\frac{\Delta}{4}$ from the left end:

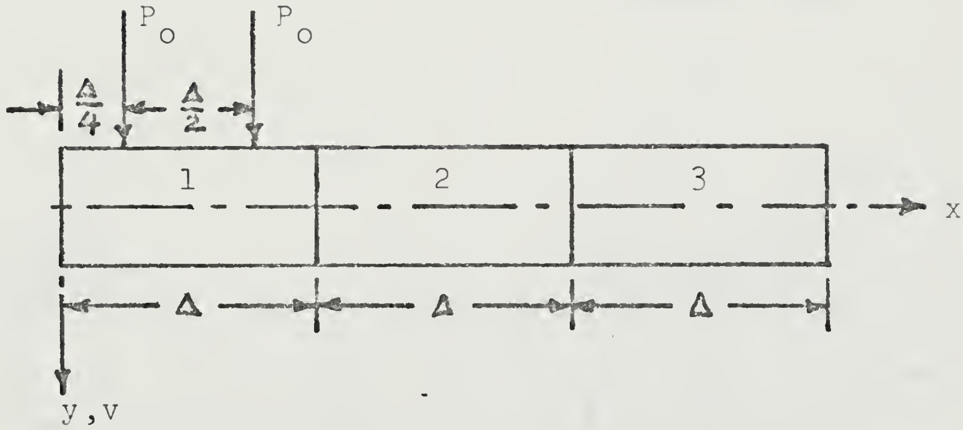


Figure 16. Three Element Beam, Two Concentrated Loads on Element One.

The element consistent load distribution vector for a concentrated load P_o a distance $\Delta/4$ from the origin is:

$$<D(\tfrac{\Delta}{4})>_i = < \frac{27}{32}, \frac{9\Delta}{64}, \frac{5}{32}, -\frac{3\Delta}{64} > \qquad \text{(C-6)}$$

In this example $<D>_2$ and $<D>_3$ are the same as in example 3. That is $<D>_2 = <D>_3 = <0,0,0,0>$. The consistent load distribution vector for element 1 is the sum:

$$<D>_1 = <D(\tfrac{\Delta}{4})> + <D(\tfrac{3\Delta}{4})>$$

Thus:

$$<D>_1 = <1, \frac{3\Delta}{16}, 1, -\frac{3\Delta}{16}>$$

Assembling the element vectors we obtain for the global vector:

$$<\bar{D}> = <1, \frac{3\Delta}{16}, 1, -\frac{3\Delta}{16}, 0,0,0,0> \qquad \text{(C-7)}$$

5. For this example let us take the following beam with concentrated loads as shown:

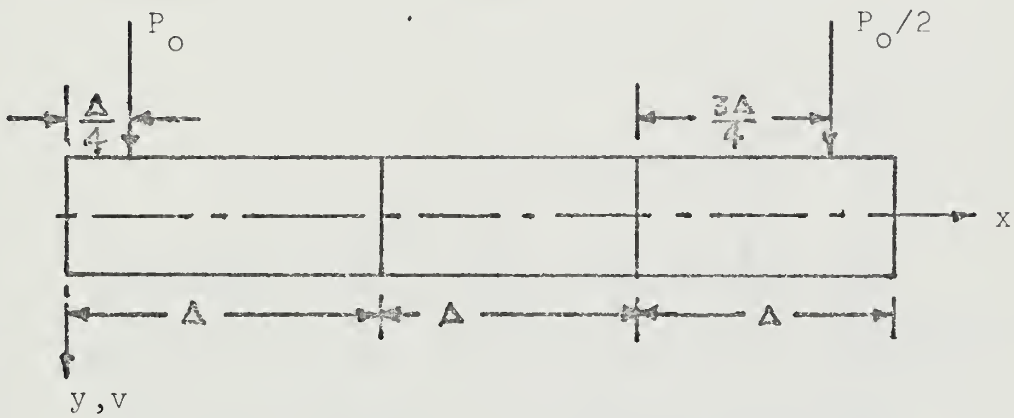


Figure 17. Three Element Beam, Concentrated Loads on Elements one and Three.

From equation (C-6) we have for $<D>_1$:

$$<D>_1 = < \frac{27}{32}, \frac{9\Delta}{64}, \frac{5}{32}, -\frac{3\Delta}{64} >$$

Element 2 has no loads thus:

$$<D>_2 = <0, 0, 0, 0>$$

Equation (C-4a) gives the element consistent load vector for a loading such as is on element 3 but of intensity P_o.

The element consistent load distribution vector for element 3 is therefore one-half of that value in (C-4a). Thus

$$\langle D \rangle_3 = \left\langle \frac{5}{64}, \frac{3\Delta}{128}, \frac{27}{64}, -\frac{9\Delta}{128} \right\rangle$$

Assembling these to obtain the global consistent load distribution vector:

$$\langle \bar{D} \rangle = \left\langle \frac{27}{32}, \frac{9\Delta}{64}, \frac{5}{32}, -\frac{3\Delta}{64}, \frac{5}{64}, \frac{3\Delta}{128}, \frac{27}{64}, -\frac{9\Delta}{128} \right\rangle \qquad \text{(C-8)}$$

6. As a final example let us take a three element beam under a triangular load as shown:

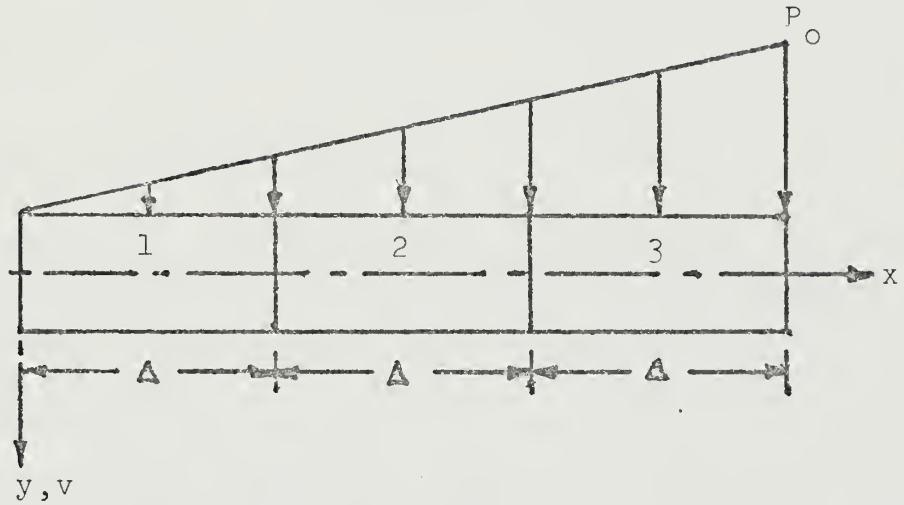


Figure 18. Three Element Beam, Triangular Load.

We first examine a general element with a basic triangular load of maximum intensity P as shown:

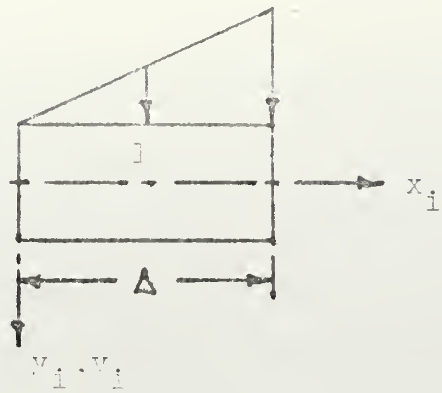


Figure 19. General Element, Triangular Load.

This distribution may be written:

$$P(x_i) = \frac{Px_i}{\Delta}$$

The element consistent load distribution vector by definition is:

$$\langle D \rangle_i = \int_0^\Delta \frac{x_i}{\Delta} \langle N(x_i) \rangle \, dx_i$$

Evaluating the integral we obtain for the basic element consistent load distribution for a triangular load, $P(x_i) = Px_i$:

$$\langle D \rangle_i = \langle \frac{3\Delta}{20}, \frac{\Delta^2}{30}, \frac{21\Delta}{60}, -\frac{\Delta^2}{20} \rangle \qquad \text{(C-9)}$$

If we depict the example problem as follows:

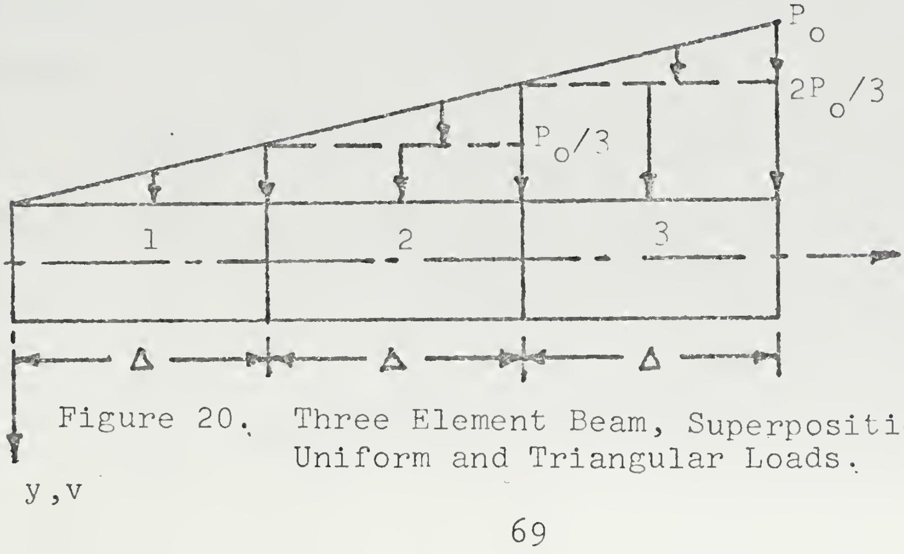


Figure 20. Three Element Beam, Superposition of Uniform and Triangular Loads.

We see that element 1 has a basic triangular load distribution of intensity $P_o/3$. Element 2 has a uniform load distribution of intensity $P_o/3$ with the triangular distribution of element 1 superposed. Element 3 similarly has this superposition but the uniform load intensity is $2P_o/3$.

From (C-9) and the relation $P = P_o/3$ we obtain:

$$<D>_1 = \frac{1}{3} <D>_i^{triang} = < \frac{\Delta}{20}, \frac{\Delta^2}{90}, \frac{7\Delta}{60}, - \frac{\Delta^2}{20} >$$

From the result for $<D>_1$ above, and from (C-1) with intensity $P_o/3$, the uniform portion of $<D>_2$ is:

$$<D>_2^{uniform} = \frac{1}{3} < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, \frac{-\Delta^2}{12} >$$

Thus:

$$<D>_2^{uniform} = < \frac{\Delta}{6}, \frac{\Delta^2}{36}, \frac{\Delta}{6}, \frac{-\Delta^2}{12} >$$

And:

$$<D>_2 = <D>_2^{triang} + <D>_2^{uniform}$$

$$<D>_2 = < \frac{13\Delta}{60}, \frac{7\Delta^2}{180}, \frac{17\Delta}{60}, - \frac{2\Delta^2}{45} >$$

The third element consistent load distribution vector is developed in a similar manner:

$$<D>_3^{uniform} = \frac{2}{3} < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \frac{\Delta}{2}, - \frac{\Delta^2}{12} >$$

$$= < \frac{\Delta}{3}, \frac{\Delta^2}{18}, \frac{\Delta}{3}, - \frac{\Delta^2}{18} >$$

And:

$$<D>_3 = <D>_3^{uniform} + <D>_3^{triangular}$$

Or:

$$\langle D \rangle_3 = \langle \frac{23\Delta}{60}, \frac{\Delta^2}{15}, \frac{9\Delta}{20}, -\frac{13\Delta^2}{180} \rangle$$

Assembling $\langle D \rangle_1$, $\langle D \rangle_2$, and $\langle D \rangle_3$ we obtain:

$$\langle \bar{D} \rangle = \frac{\Delta}{20}, \frac{\Delta^2}{90}, \frac{\Delta}{3}, \frac{\Delta^2}{45}, \frac{2\Delta}{3}, \frac{\Delta^2}{45}, \frac{9\Delta}{20}, -\frac{13\Delta^2}{180} \rangle \qquad \text{(C-10)}$$

APPENDIX D

FINITE ELEMENT OPTIMIZATION EXAMPLES

All of the following examples consider the approximate structural optimization of a rectangular cross-section beam of uniform height and fixed volume. The physical parameters are:

a) beam length = 120 in

b) beam height = 6 in

c) beam volume = 2200 in^3

d) modulus of elasticity = 30×10^6 lb/in^2

e) yield strength = 50×10^3 lb/in^2

A three element approximation is used throughout.

1. Representative problem set-up

As a representative problem consider a simply supported beam under a uniform load. The beam is depicted in Figure 21. NI = 3.

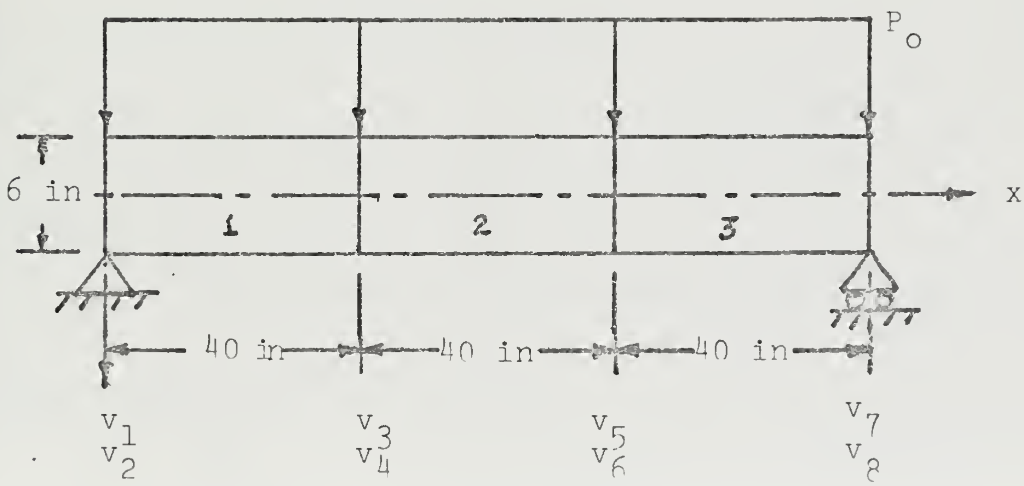


Figure 21. Three Element Problem, Uniform Load.

Since the beam is simply supported the boundary conditions are:

$$v_1 = 0$$

$$v_7 = 0$$

Therefore:

$$IBC(1) = 1 \qquad BC(1) = 0.0$$

$$IBC(2) = 7 \qquad BC(2) = 0.0$$

$$NBC = 2$$

The total number of parameters are:

$$NT = 3 \times 3 + 3 = 12$$

The total number of unknowns are:

$$NU = 12 - 2 = 10$$

From Appendix C, the consistent load distribution vector for a uniform load on three elements is:

$$CLOAD = < \frac{\Delta}{2}, \frac{\Delta^2}{12}, \Delta, 0, \Delta, 0, \frac{\Delta}{2}, -\frac{\Delta^2}{12} >$$

Since $\Delta = L/3 = 40$ in, CLOAD becomes:

$$< 20, 133.333, 40, 0, 40, 0, 20, -133.33 >$$

The components of the x vector are:

$$x = < v_2, v_3, v_4, v_5, v_6, v_8, A_1, A_2, A_3, P_o >$$

We guess values for the x vector:

$v_2 = .03$ $A_1 = 15$ in^2 $P_o = 760$ lb/in

$v_3 = .9$ in $A_2 = 28$ in^2

$v_4 = .01$ $A_3 = 15$ in^2

$v_5 = .9$ in

$v_6 = -.01$ $v_8 = -.03$

We select values for MAXIT, NUMSIG, and IPRINT. Let us require 50 iterations maximum.

MAXIT = 50

We desire four significant digits.

NUMSIG = 4

We desire the x value printed at each iteration.

IPRINT = 0

We now can utilize the program. The calling program is:

```
DIMENSION X(10)
CALL OPBEAM(10,50,4,0,X)
STOP
END
```

We insert the following cards in their proper locations in FCNLST.

```
DIMENSION IBC(2),BC(2),CLOAD(8),G(12),H(12),
          V(8),A(3)
DATA NU/10/,NBC/2/,NI/3/,NT/12/
```

The first data card is the first six components of the x vector in E12.5 format. The second is the last four components of the x vector in E12.5 format. The third card is the list of physical parameters BLENG,E,HI,VOL,SIGYP in E12.5 format. The fourth card is the first component of IBC and the corresponding component of BC (I10,E12.5). The fifth card is the same as the fourth but for the second components of the vectors.

The last two cards contain the eight components of the consistent load distribution vector in 6E12.5 format. The data deck is thus:

0.3E-1	0.9E0	0.1E-1	0.9E0	-0.1E-1	-0.3E-1
1.5E1	2.8E1	1.5E1	7.6E2		
1.2E1	3.0E7	6.0E0	2.2E3	5.0E4	
1	0.0E0				
2	0.0E0				
2.0E1	1.33333E2	4.0E1	0.0E0	4.0E1	0.0E0
2.0E1	-1.33333E2				

This program was run and the following solution was returned after four iterations.

v_2 = .0310 $\quad A_1$ = 15.01 in^2 $\quad P_o$ = 720.6 lb/in
v_3 = .9566 in $\quad A_2$ = 24.98 in^2
v_4 = .0111 $\quad A_3$ = 15.01 in^2
v_5 = .9566 in
v_6 = -.0111
v_8 = -.0310

We are interested in mainly the areas and maximum load results. If the areas are divided by the height (6 in) we obtain the width. Since the beam is symmetric about its centerline we may plot one half of the beam to depict its shape. Figure 22 is a plot of the three finite element approximation with the exact solution superimposed.

2. Other example problems

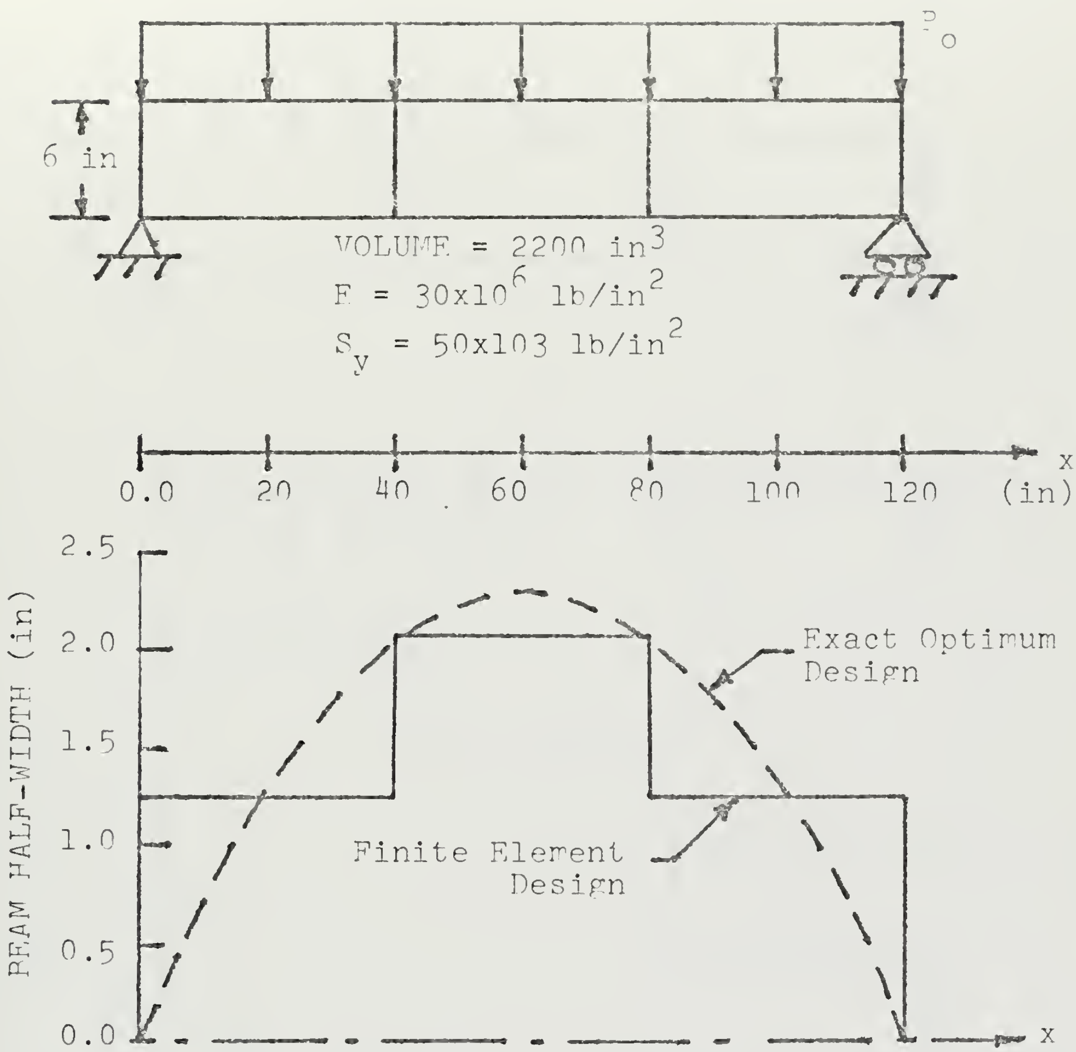
The problems of a simply supported beam under concentrated loads at various locations were also solved with

the computer program. The results for these problems are shown in Figures 23, 24, 25, and 26. Corresponding loads to the right of the beam midpoint were checked and the beam shapes obtained were the inverse of those shown, i.e., the half-width values for the end elements were interchanged.

The solution of a simply supported beam under a triangular load distribution is shown in Figure 27.

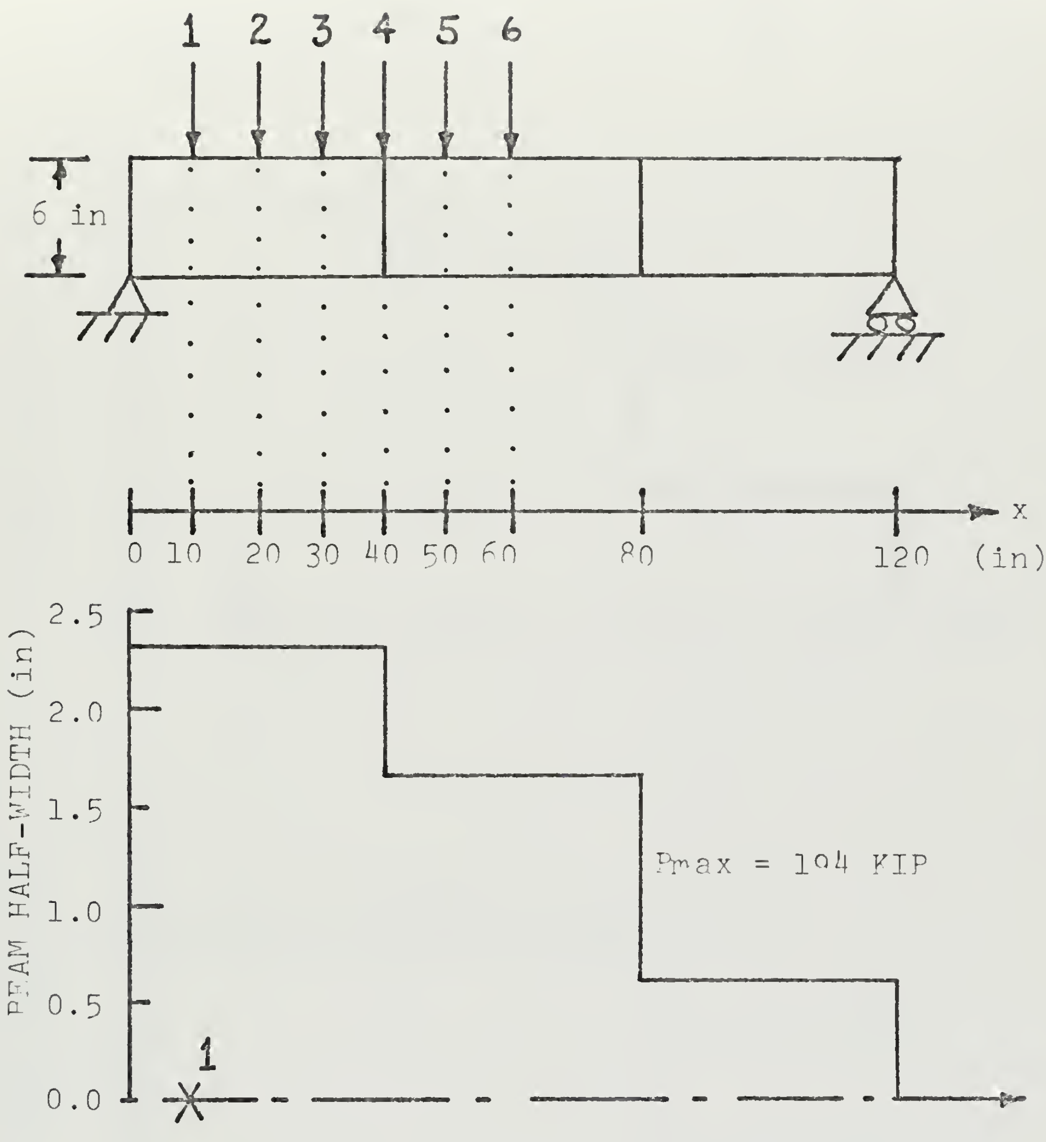
Finally the problem of a cantilever beam under a uniform load was solved. This is shown in Figure 28.

Optimum beam shapes for the simply supported beams under 1) a uniform load and 2) a concentrated load at the center and also for the cantilever beam under a uniform load are shown in [11]. These exact optimum shapes are superimposed for comparison in Figures 22, 26, and 28.



MAXIMUM LOAD, FINITE ELEMENT SOLUTION...721 lb/in
MAXIMUM LOAD, EXACT SOLUTION............764 lb/in
MAXIMUM LOAD, UNIFORM CROSS-SECTION.....510 lb/in

Figure 22. Three Element Optimum Design for a Simply supported Beam under Uniform Load.



Loads at points 1 through 6 apply in Figures 23 - 26. The beams shown in these figures have the parameters:

$$V_o = 2200 \text{ in}^3$$
$$E = 30\times10^6 \text{ lb/in}^2$$
$$S_y = 50\times10^3 \text{ lb/in}^2$$

Figure 23. Three Element Optimum Design for a Simply Supported Beam under a Concentrated Load Applied at Point 1.

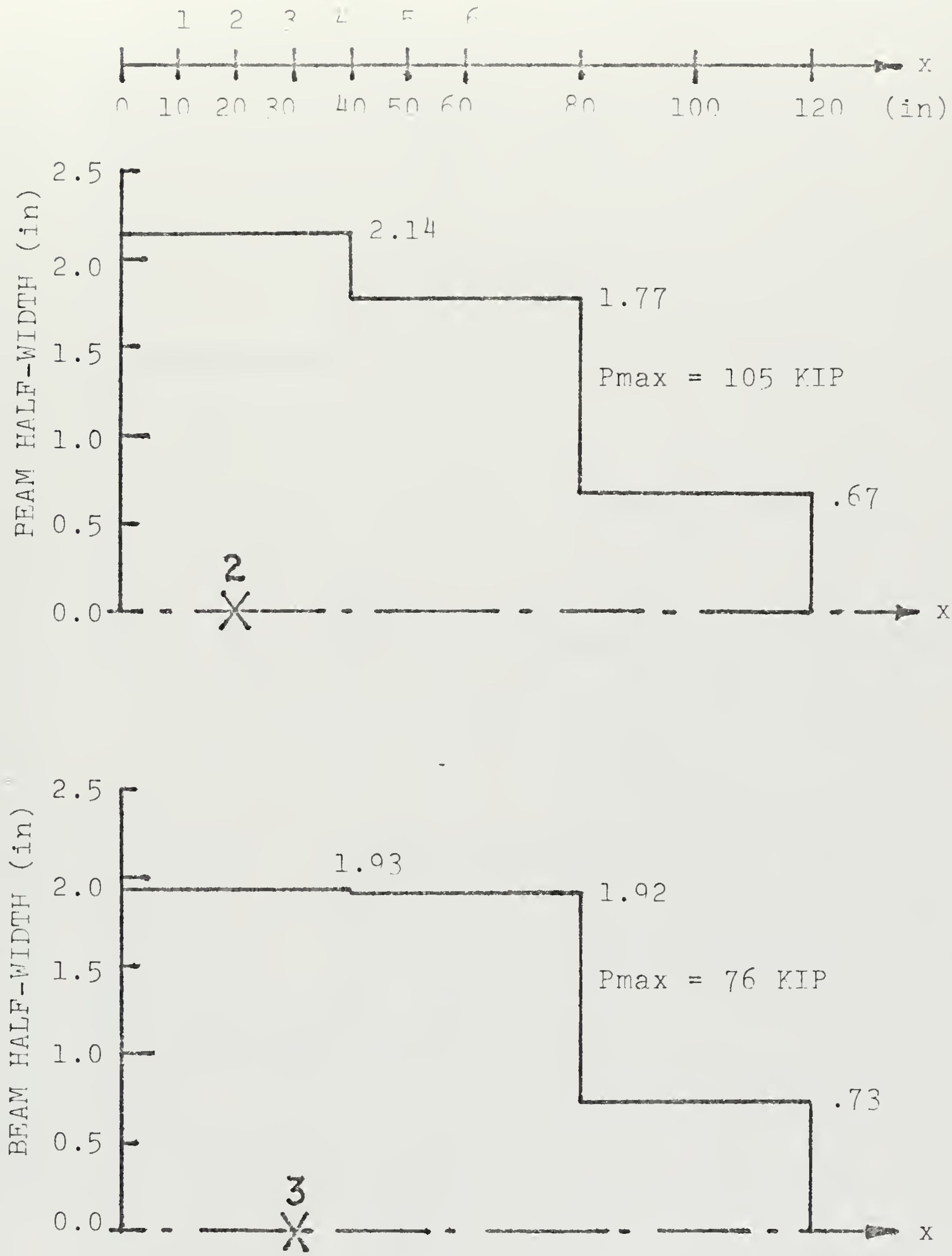


Figure 24. Three Element Optimum Beam Designs for Simply Supported Beams under Concentrated Loads at Points 2 and 3.

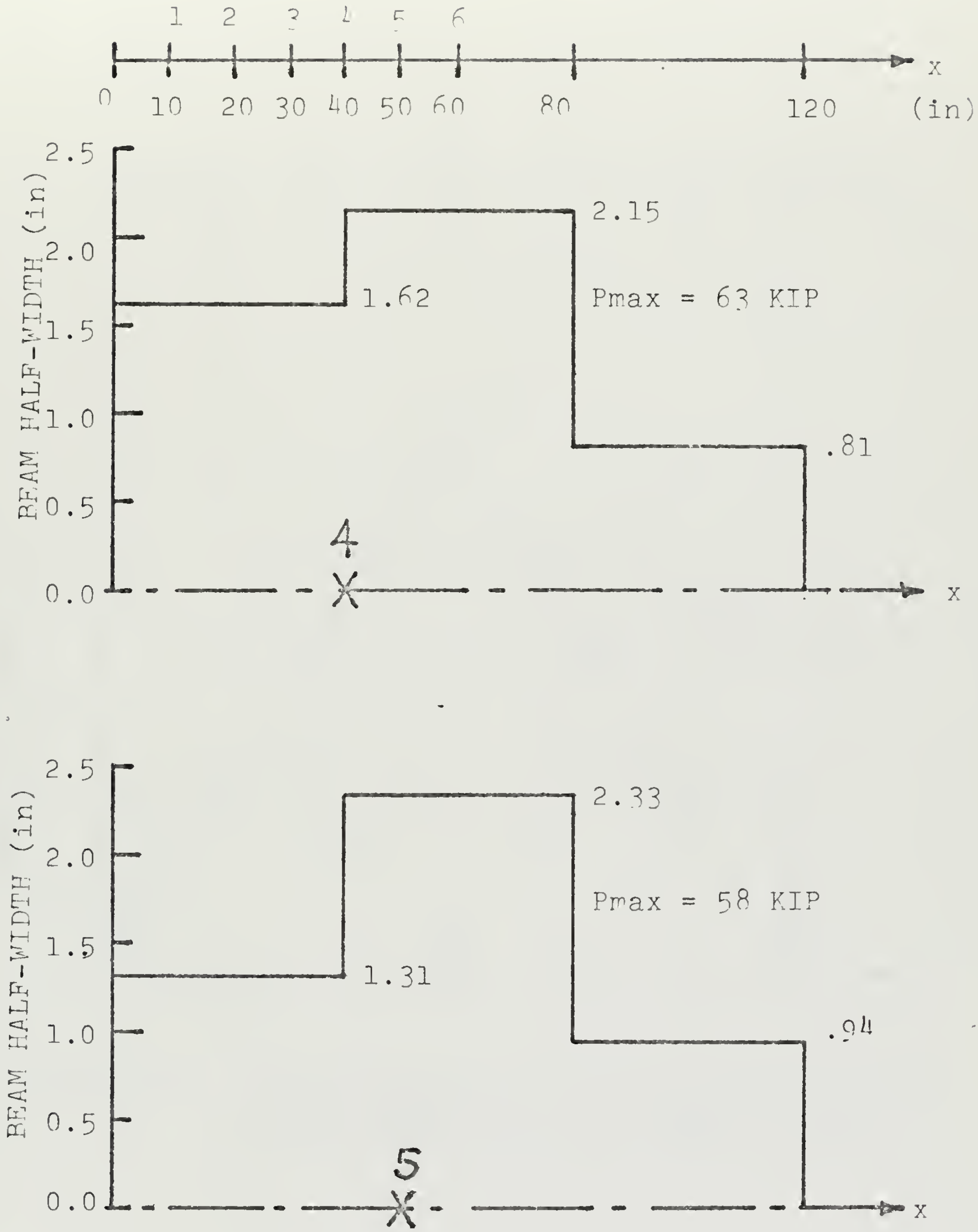


Figure 25. Three Element Optimum Beam Designs for Simply Supported Beams under Concentrated Loads Applied at Points 4 and 5.

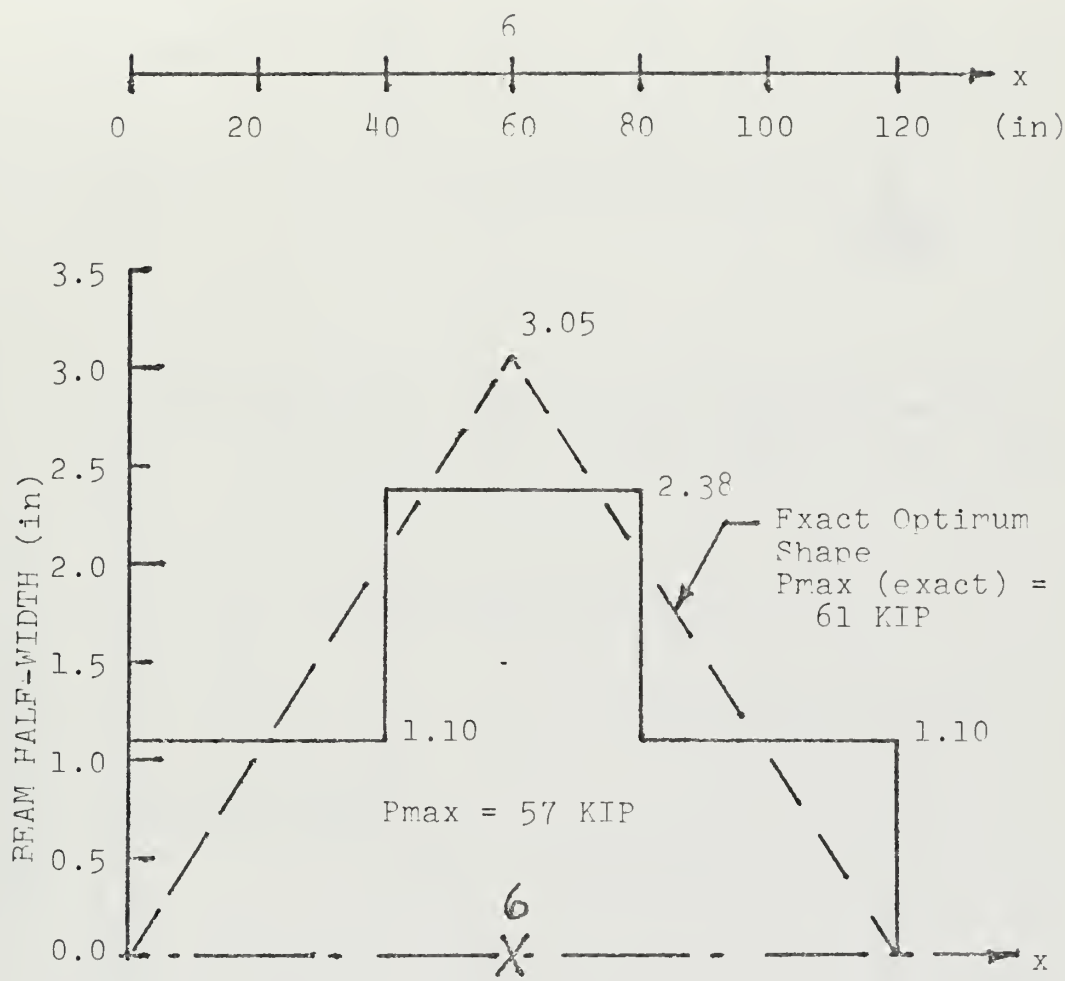


Figure 26. Three Element Optimum Beam Design for a Simply Supported Beam under a Concentrated Load Applied at Point 6.

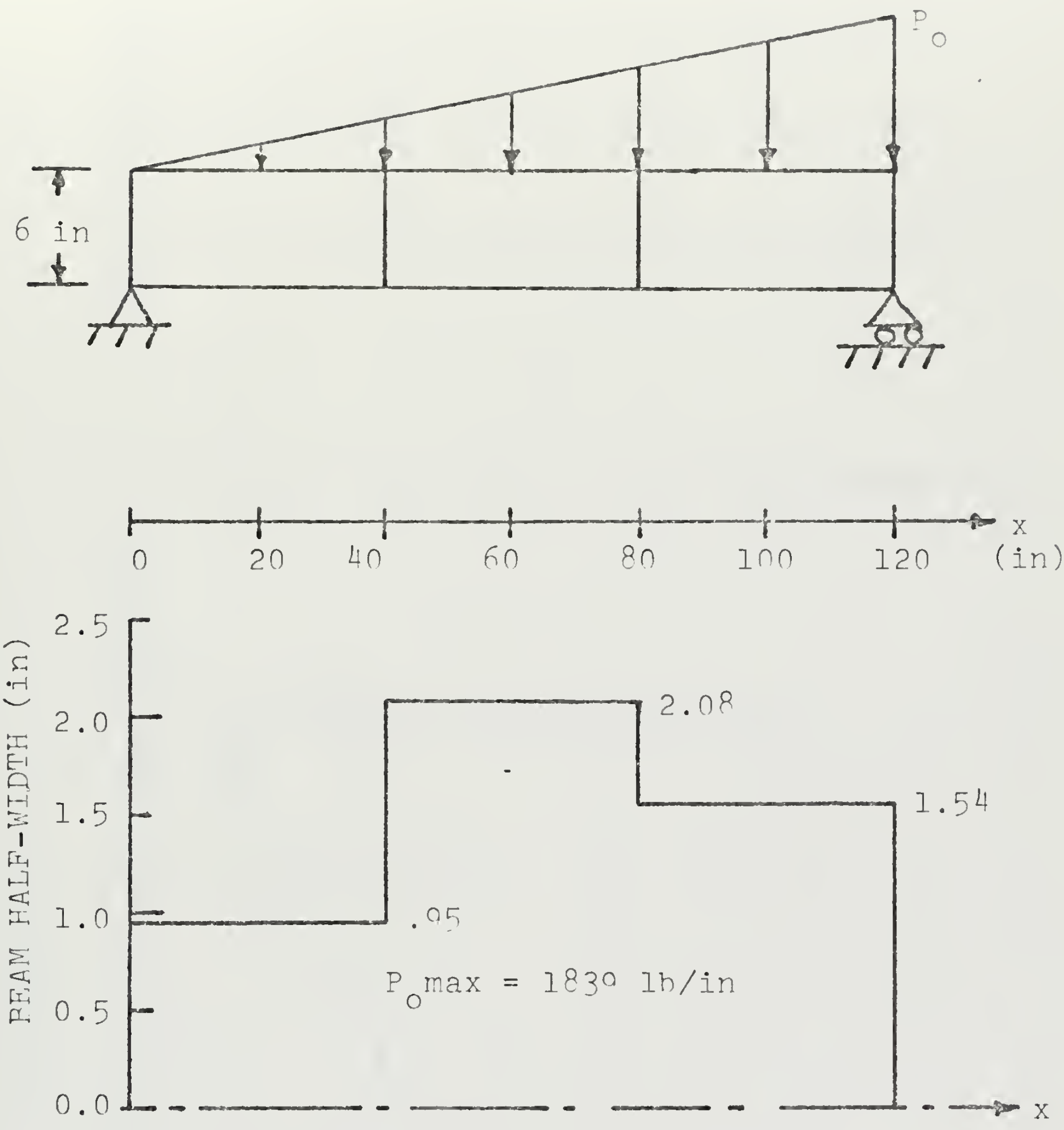


Figure 27. Three Element Optimum Beam Design for a Simply Supported Beam under a Triangular Load Distribution.

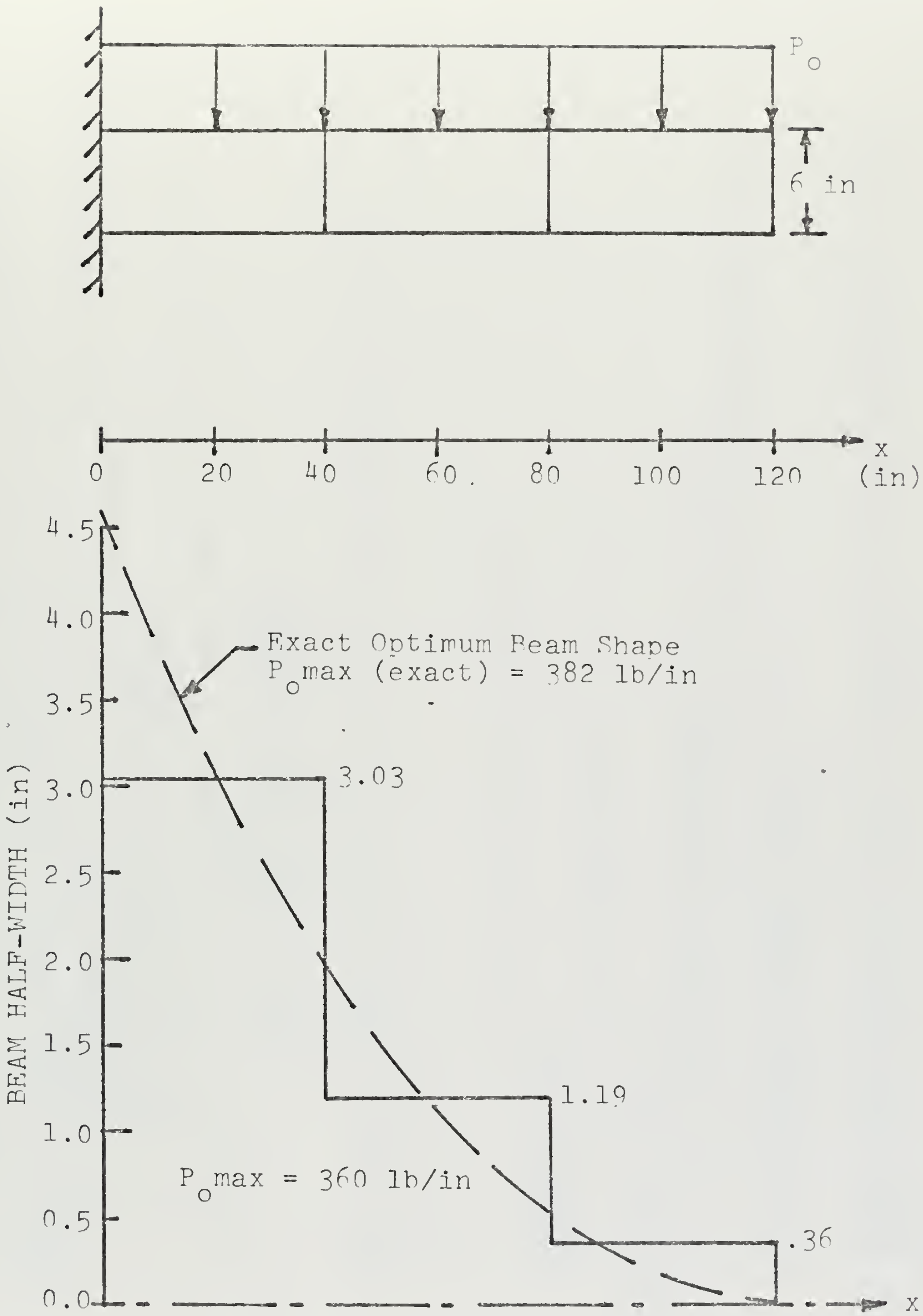


Figure 28. Three Element Optimum Beam Design for a Cantilever Beam under a Uniform Load Distribution.

```
C      SUBROUTINE OPBEAM
C      ......................................................................
C
C      THIS SUBROUTINE READS IN THE STARTING GUESS VECTOR FOR SUBROUTINE
C      NLNSYS.  IT ALSO OUTPUTS THE RESULTS FROM NLNSYS, WHICH MAY BE ONE
C      OF THE FOLLOWING:
C            1)  A SINGULARITY OCCURRED AT THE PRINTED X VALUE; THE
C                MESSAGE IS GIVEN.
C            2)  THE ITERATION LIMIT WAS REACHED, THEREFORE THE PRINTED X
C                VALUE MAY NOT BE A SOLUTION.
C            3)  THE SOLUTION VECTOR IS PRINTED.
C
C      PARAMETERS:  NU......THE NUMBER OF EQUATIONS IN THE SYSTEM TO BE
C                           SOLVED.
C                   MAXIT...DESCRIBED IN NLNSYS.
C                   NUMSIG..DESCRIBED IN NLNSYS
C                   IPRINT..DESCRIBED IN NLNSYS
C                   X.......DESCRIBED IN NLNSYS
C
C      DATA CARDS:  OPBEAM REQUIRES THAT THE FIRST DATA CARDS IN THE DATA
C      DECK BE THE INITIAL GUESS VECTOR FOR X.  THE FORMAT IS 6E12.5.
C
C      ......................................................................
C

      SUBROUTINE OPBEAM(NU,MAXIT,NUMSIG,IPRINT,X)
      DIMENSION X(1)
      EXTERNAL FCNLST
C
C     READ IN THE INITIAL GUESS VECTOR
   90 READ(5,100) (X(I),I=1,NU)
  100 FORMAT (6E12.5)
C
C     START THE SOLUTION TO THE PROBLEM
      CALL NLNSYS(NU,MAXIT,NUMSIG,ISING,IPRINT,FCNLST,X)
C
C     IF A SINGULARITY HAS BEEN GENERATED, PRINT THE MESSAGE WITH THE
C     CORRESPONDING X VECTOR VALUE.
      IF (ISING.EQ.1) WRITE(6,110)
C
C     PRINT THE FINAL X VECTOR
  105 WRITE(6,120) (I,X(I),I=1,NU)
  110 FORMAT (5X,'SYSTEM CREATED A SINGULARITY NEAR THE FOLLOWING X VALU
     1E')
  120 FORMAT (/2X,5(2X,'X(',I4,')= ',E12.5))
      RETURN
      END
```

```
C     SUBROUTINE FCNLST
C
C     ..................................................................
C
C     THIS SUBROUTINE GENERATES A SYSTEM OF NON-LINEAR EQUATIONS WHICH
C     DESCRIBES THE FINITE ELEMENT MODEL APPROXIMATING THE OPTIMUM
C     STRENGTH DESIGN OF A BEAM WITH RECTANGULAR CROSS-SECTION OF A
C     UNIFORM FIXED HEIGHT, FIXED LENGTH, AND FIXED VOLUME UNDER A GIVEN
C     LOAD DISTRIBUTION.  IT PROVIDES THE VALUE OF A SELECTED FUNCTION
C     EVALUATED AT AN ESTIMATED SOLUTION VECTOR TO SUBROUTINE NLNSYS.
C
C     DESCRIPTION OF PARAMETERS:
C          X.......A SOLUTION VECTOR ESTIMATE PROVIDED BY NLNSYS
C          F.......THE VALUE OF THE K'TH FUNCTION AT THE X VALUE
C          K.......AS ABOVE
C          INSIG..AN INPUT SIGNAL PROVIDED BY NLNSYS SUCH THAT:
C                  INSIG=0 THE FIRST TIME FCNLST IS CALLED
C                  INSIG=1 ALL OTHER TIMES
C
C     STATEMENTS REQUIRED BY USER:
C          THE USER MUST PROVIDE THE FOLLOWING TWO STATEMENTS AS INDIC-
C          ATED IN THE PROGRAM:
C          1)  THE DIMENSION STATEMENT
C
C     DIMENSION IBC(NBC),BC(NBC),CLOAD(NV),G(NT),H(NT),V(NV),A(NI)
C
C              THE VALUES IN PARENTHESES ARE NUMERICAL AND ARE:
C                  NBC= NUMBER OF BOUNDARY CONDITIONS
C                  NI= NUMBER OF ELEMENTS
C                  NV= NUMBER OF COMPONENTS IN THE VECTOR V
C                            NV=(2*NI)+2
C                  NT= TOTAL NUMBER OF PARAMETERS
C                            NT=(3*NI)+3
C
C          2)  THE DATA STATEMENT
C
C     DATA NU/*/,NBC/*/,NI/*/,NT/*/
C
C              THE ASTERISKS REPRESENT NUMERICAL VALUES AND ARE:
C                  NBC AND NI ARE AS ABOVE
C                  NU= NUMBER OF UNKNOWNS
C                            NU=NT-NBC
C
C     DATA CARDS REQUIRED:
C     THE FOLLOWING DATA CARDS ARE REQUIRED.  THEY ARE INSERTED IN ORDER
C     FOLLOWING THE DATA CARDS WHICH CONTAIN THE INITIAL X GUESS.
C
C          A)  THE PHYSICAL CONSTANT DATA CARD:
C              THIS CARD LISTS THE PROBLEM'S PHYSICAL CONSTANTS IN
```

```
C              5E12.5 FORMAT IN THE FOLLOWING ORDER:
C                  BLENG= BEAM LENGTH (IN)
C                  E= MODULUS OF ELASTICITY (LB/IN2)
C                  HI= BEAM HEIGHT (IN)
C                  VOL= BEAM VOLUME (IN3)
C                  SIGYP= YIELD STRESS (LB/IN2)
C
C          B)  THE BOUNDARY CONDITION DATA CARDS:
C              THERE IS ONE CARD FOR EACH BOUNDARY CONDITION. THEY ARE
C              ARRANGED IN INCREASING ORDER AND EACH CONTAINS IN I10,
C              E12.5 FORMAT:
C                  1) THE SUBSCRIPT OF THE COMPONENT OF THE DISPLACEMENT
C                     VECTOR WHICH IS A BOUNDARY CONDITION (I10).
C                  2) THE VALUE OF THAT BOUNDARY CONDITION (E12.5).
C              THERE IS A TOTAL OF NBC BOUNDARY CONDITION CARDS.
C
C          C)  THE CONSISTENT LOAD DISTRIBUTION VECTOR IN 6E12.5 FORMAT.
C
C      ..................................................................
C

      SUBROUTINE FCNLST(X,F,K,INSIG)
      DIMENSION X(1),STIFF(4,4)
C
C     THE FOLLOWING STATEMENT IS SUPPLIED BY THE USER WITH NUMERIC
C     VALUES SUBSTITUTED FOR *.
      DIMENSION IBC(*),BC(*),CLOAD(*),G(*),H(*),V(*),A(*)
C
      REAL*4 LAMB
C
C     THE FOLLOWING STATEMENT IS SUPPLIED BY THE USER WITH NUMERIC
C     VALUES SUBSTITUTED FOR *.
      DATA NU/*/,NBC/*/,NI/*/,NT/*/
C
      IF ((INSIG.NE.1).OR.(K.NE.1)) GO TO 100
C     READ IN THE DATA THE FIRST TIME THROUGH THE SUBROUTINE
C
C     READ IN THE PROBLEM PHYSICAL CONSTANTS
      READ(5,10) BLENG,E,HI,VOL,SIGYP
   10 FORMAT (5E12.5)
C
C     CALCULATE NEEDED CONSTANTS
      C=HI**2/12.
      EC=E*C
      LAMB=SIGYP**2/(6.*E)
      N=NI*2+2
      NA=NI
```

```
      FLOAT=NI
C
C     CALCULATE ELEMENT LENGTH
      ELENG=BLENG/FLOAT
C
C     READ AND WRITE THE BOUNDARY CONDITIONS
      DO 12 I=1,NBC
      READ(5,20) IBC(I),BC(I)
   20 FORMAT (I10,E12.5)
   12 CONTINUE
      WRITE(6,25) (IBC(I),BC(I),I=1,NBC)
   25 FORMAT (///25X,'BOUNDARY CONDITIONS ARE :'/(30X,'V(',I4,')= ',E12.
     15))
C
C     READ THE CONSISTENT LOAD DISTRIBUTION VECTOR
      READ(5,40) (CLOAD(I),I=1,N)
   40 FORMAT (6E12.5)
C
C     WRITE THE PROBLEM PHYSICAL CONSTANTS
      WRITE(6,50) BLENG,NI,HI,E,VOL,SIGYP
   50 FORMAT (//5X,'BEAM LENGTH= ',F10.2,' IN.'/5X,'NUMBER OF ELEMENTS=
     1',I4/5X,'BEAM HEIGHT= ',F10.4,' IN.'/5X,'MODULUS OF ELASTICITY= ',
     2E12.5,' PSI'/5X,'BEAM VOLUME= ',E12.5,' CUBIC INCHES'/5X,'YIELD ST
     3RESS= ',E12.5,'PSI'//)
C
C     WRITE THE CONSISTENT LOAD DISTRIBUTION VECTOR
      WRITE(6,60) (CLOAD(I),I=1,N)
   60 FORMAT (///,5X,'THE COMPONENTS OF THE CONSISTENT LOAD VECTOR ARE :
     1'//(5X,6(E12.5,5X)/))
C
C     CALCULATE THE MODIFIED ELEMENT STIFFNESS MATRIX
      STIFF(1,1)=12./ELENG**3
      STIFF(1,2)=6./ELENG**2
      STIFF(1,3)=-STIFF(1,1)
      STIFF(1,4)=STIFF(1,2)
      STIFF(2,1)=STIFF(1,2)
      STIFF(2,2)=4./ELENG
      STIFF(2,3)=-STIFF(1,2)
      STIFF(2,4)=2./ELENG
      STIFF(3,1)=STIFF(1,3)
      STIFF(3,2)=STIFF(2,3)
      STIFF(3,3)=STIFF(1,1)
      STIFF(3,4)=STIFF(2,3)
      STIFF(4,1)=STIFF(1,4)
      STIFF(4,2)=STIFF(2,4)
      STIFF(4,3)=STIFF(3,4)
      STIFF(4,4)=STIFF(2,2)
C
```

```
C     INSERT BOUNDARY CONDITIONS AND ESTIMATED SLOPES AND DEFLECTIONS
C     INTO THEIR PROPER PLACES IN THE VECTOR V OF SLOPES AND DISPL'M'TS.
  100 DO 130 I=1,NBC
      KJ=IBC(I)
      DO 120 J=KJ,N
      IF (J.EQ.IBC(I)) GO TO 110
      V(J)=X(J-I)
      GO TO 120
  110 V(J)=BC(I)
  120 CONTINUE
  130 CONTINUE
C
C
      L=IBC(1)-1
      IF (IBC(1).NE.1) GO TO 140
      GO TO 160
C
C
  140 DO 150 J=1,L
      V(J)=X(J)
  150 CONTINUE
C
C     INSERT AREA ESTIMATES FROM X VECTOR INTO A VECTOR OF AREAS A.
  160 DO 170 I=1,NA
      A(I)=X(N-NBC+I)
  170 CONTINUE
C
C     CALL THE LAST COMPONENT OF THE SOLUTION ESTIMATE PZERO, THE LOAD
      PZERO=X(NU)
C
C     GENERATE THE VECTOR OF GENERAL FUNCTIONS, G, WHICH INCLUDES THE
C     FUNCTIONS DERIVED FROM THE BOUNDARY NODES.
  179 DO 190 I=1,2
      SUM=0.
      DO 180 J=1,4
      PROD=STIFF(I,J)*V(J)
      SUM=SUM+PROD
  180 CONTINUE
C
C
      G(I)=A(1)*SUM-PZERO*CLOAD(I)/EC
  190 CONTINUE
C
C
      DO 210 I=3,4
      SUM=0.
      DO 200 J=1,4
      PROD=STIFF(I,J)*V(J+N-4)
```

```
      SUM=SUM+PROD
  200 CONTINUE
      NP=N+I-4
      G(NP)=A((N-2)/2)*SUM-PZERO*CLOAD(NP)/EC
  210 CONTINUE
C
C
      LK=N-2
      DO 230 I=4,LK,2
      SUM1=0.
      SUM2=0.
      SUM3=0.
      SUM4=0.
      DO 220 J=1,4
      PROD1=STIFF(3,J)*V(J+I-4)
      PROD2=STIFF(1,J)*V(J+I-2)
      PROD3=STIFF(4,J)*V(J+I-4)
      PROD4=STIFF(2,J)*V(J+I-2)
      SUM1=SUM1+PROD1
      SUM2=SUM2+PROD2
      SUM3=SUM3+PROD3
      SUM4=SUM4+PROD4
  220 CONTINUE
      IM=I-1
      G(IM)=(A((I-2)/2)*SUM1+A(I/2)*SUM2)-PZERO*CLOAD(IM)/EC
      G(I)=(A((I-2)/2)*SUM3+A(I/2)*SUM4)-PZERO*CLOAD(I)/EC
  230 CONTINUE
C
C
      DO 260 M=1,NA
      SUM=0.
      DO 250 I=1,4
      DO 240 J=1,4
      PROD=STIFF(I,J)*V(J+2*M-2)*V(I+2*M-2)
      SUM=SUM+PROD
  240 CONTINUE
  250 CONTINUE
      NPM=N+M
      G(NPM)=SUM*EC-(2.*LAMB*ELENG)
  260 CONTINUE
C
C
      SUM=0.
      DO 270 I=1,NA
      SUM=SUM+A(I)
  270 CONTINUE
      G(NT)=ELENG*SUM-VOL
C
```

```
C      REMOVE THE FUNCTIONS DERIVED FROM BOUNDARY CONDITIONS AND FORM THE
C      REDUCED VECTOR, H, OF ADMISSIBLE FUNCTIONS
       IF (IBC(1).EQ.1) GO TO 290
       DO 280 J=1,L
       H(J)=G(J)
  280  CONTINUE
  290  DO 310 I=1,NBC
       KJ=IBC(I)
       DO 300 J=KJ,NT
       IF (J.EQ.KJ) GO TO 300
       JJ=J-I
       H(JJ)=G(J)
  300  CONTINUE
  310  CONTINUE
C
C      RETURN THE K'TH ADMISSIBLE FUNCTION VALUE TO NLNSYS
       F=H(K)
  320  RETURN
       END
```

```
C
C     ..................................................................
C
C        SUBROUTINE NLNSYS
C
C        IDENTIFICATION
C           TITLE      - NLNSYS    (SOLUTION OF SIMULTANEOUS NON-LINEAR
C                                   EQUATIONS).
C           ID         - C4-NPG-NLNSYS (FORTRAN IV G).
C           CATEGORY   -MATHEMATICAL SUBROUTINE
C           PROGRAMMER - D. D. PURCELL
C           DATE       - FEB. 1969
C
C        PURPOSE
C           TO SOLVE A SYSTEM OF N NON-LINEAR EQUATIONS.  THESE MUST BE
C           SUPPLIED BY THE USER IN A SUBROUTINE REFERENCED BY THE MAIN-
C           LINE CALLING ROUTINE.
C
C        USAGE
C           CALL NLNSYS(N,MAXIT,NUMSIG,ISING,IPRINT,EVALUT,X)
C
C        DESCRIPTION OF PARAMETERS
C           N      - THE NUMBER OF EQUATIONS.
C           MAXIT  - THE ITERATION LIMIT.
C           NUMSIG - THE NUMBER OF SIGNIFICANT PLACES REQUIRED IN THE
C                    SOLUTION.  IT SHOULD BE LESS THAN OR EQUAL TO 7,
C                    WHICH IS THE LIMIT WITH SINGLE PRECISION.
C           ISING  - A RETURN PARAMETER SUCH THAT:
C                    1. ISING IS 0 IF THE STARTING GUESS TOGETHER WITH
C                       THE SYSTEM OF EQUATIONS DOES NOT GENERATE A
C                       SINGULARITY DURING THE PROCESS.
C                    2. ISING IS 1 IF THE STARTING GUESS TOGETHER WITH
C                       THE SYSTEM OF EQUATIONS DOES GENERATE A SING-
C                       ULARITY DURING THE PROCESS.
C           IPRINT - USER SETS THIS TO ZERO IF HE WANTS THE SUCCESSIVE
C                    ITERATIVE APPROXIMATIONS TO BE PRINTED (ON UNIT 6).
C                    ANYTHING ELSE WILL OVERIDE THIS PRINT-OUT.
C           EVALUT - THE SUBROUTINE CREATED BY THE USER WHICH SPECIFIES
C                    THE NON-LINEAR SYSTEM AND RETURNS THE VALUE OF A
C                    PARTICULAR MEMBER FOR SOME X VECTOR, ITS FORM AND
C                    PARAMETERS ARE SUBSEQUENTLY EXPLAINED.
C              SUBROUTINE EVALUT(X,F,K,INSIG)    WHERE X IS A SOLUTION
C                                                VECTOR ESTIMATE GENER-
C                                                ATED BY NLNSYS, AND F
C                                                IS THE VALUE OF THE KTH
C                                                FUNCTION FOR THAT X
C                                                VALUE. INSIG IS AN IN-
C                                                PUT SIGNAL WHICH IS 0
```

```
C                                                            THE FIRST TIME THAT
C                                                            EVALUT IS CALLED AND 1
C                                                            FOR ALL ADDITIONAL
C                                                            TIMES. IT ALLOWS INPUT
C                                                            DATA TO BE READ INTO
C                                                            EVALUT.
C           X      - THE STARTING GUESS VECTOR AT CALLING TIME AND THE  NLNS0490
C                    SOLUTION WHEN CONTROL IS RETURNED TO THE MAINLINE. NLNS0500
C        EXAMPLE                                                        NLNS0510
C           DIMENSION X(2)                                              NLNS0520
C           EXTERNAL EVALUT                                             NLNS0530
C           X(1)=1.0                                                    NLNS0540
C           X(2)=4.0                                                    NLNS0550
C           CALL NLNSYS(2,10,4,ISING,0,EVALUT,X)                        NLNS0560
C           IF (ISING.EQ.1) WRITE (6,2000)                              NLNS0570
C      2000 FORMAT(1H1,55HSYSTEM CREATED A SINGULARITY NEAR THE FOLLOWINNLNS0580
C          1G X VALUE)                                                  NLNS0590
C           WRITE (6,1000) X(1),X(2)                                    NLNS0600
C      1000 FORMAT(1H1,4HX1 =,F10.7,10X,4HX2 =,F10.7)                   NLNS0610
C           END                                                         NLNS0620
C                                                                       NLNS0630
C     SUBROUTINE EVALUT(X,F,K,INSIG)
C              THE PARAMETERS ARE RESPECTIVELY - SOLUTION VECTOR GUESS, NLNS0650
C              THE RETURN PARAMETER REPRESENTING THE VALUE OF THIS FUNC-NLNS0660
C              TION FOR THIS APPROXIMATION, AND NUMBER OF FUNCTION TO   NLNS0670
C              BE RETURNED.                                             NLNS0680
C           DIMENSION X(2)                                              NLNS0690
C           IF (K.EQ.1) GO TO 1                                         NLNS0700
C           GO TO 2                                                     NLNS0710
C         1 F=2.71828183*(.920422528*(EXP(2.*X(1)-1.)-1.)+X(2)/         NLNS0720
C          1  3.14159265-2.*X(1)                                        NLNS0730
C           GO TO 3                                                     NLNS0740
C         2 F=.5*SIN(X(1)*X(2))-X(2)/12.5663706-X(1)/2.                 NLNS0750
C         3 RETURN                                                      NLNS0760
C           END                                                         NLNS0770
C                                                                       NLNS0780
C         SPACE REQUIRED                                                NLNS0790
C           NLNSYS PLUS BAKSUB REQUIRE 3400 BYTES AND THE SUPPORT SYS-  NLNS0800
C           TEMS ROUTINES PLUS BUFFERS REQUIRE ABOUT 43,000 BYTES.      NLNS0810
C                                                                       NLNS0820
C         CAUTIONS TO USER                                              NLNS0830
C           THE STARTING GUESS MUST BE WITHIN THE REGION NECESSARY FOR  NLNS0840
C           CONVERGENCE OR THE PROCESS WILL 'BLOW UP'.  THIS REGION HAS NLNS0850
C           NO EASY DEFINITION AND IT MAY BE NECESSARY TO TRY MORE THAN NLNS0860
C           ONE STARTING GUESS.  ALSO, THE USER SUBROUTINE MUST BE TYPEDNLNS0870
C           'EXTERNAL' IN THE CALLING PROGRAM.                          NLNS0880
C         EQUIPMENT CONFIGURATION                                       NLNS0890
C           IBM 360                                                     NLNS0900
```

```
C        SUBROUTINE AND FUNCTION SUBPROGRAMS REQUIRED                   NLNS0910
C           EVALUT (USER WRITTEN AND USER NAMED)                        NLNS0920
C           BAKSUB (SUPPLIED WITH NLNSYS)                               NLNS0930
C           STANDARD SYSTEMS SUBPROGRAMS.                               NLNS0940
C                                                                       NLNS0950
C        REMARKS                                                        NLNS0960
C              THIS PROCEDURE SOLVES A SYSTEM OF N SIMULTANEOUS NON-    NLNS0970
C           LINEAR EQUATIONS.  THE METHOD IS ROUGHLY QUADRATICALLY CON- NLNS0980
C           VERGENT AND REQUIRES ONLY ((N**2/2)+(3*N/2)) FUNCTION       NLNS0990
C           EVALUATIONS PER ITERATIVE STEP AS COMPARED WITH (N**2+N)    NLNS1000
C           EVALUATIONS FOR NEWTON'S METHOD.  THIS RESULTS IN A SAVINGS NLNS1010
C           OF COMPUTATIONAL EFFORT FOR SUFFICIENTLY COMPLICATED FUNC-  NLNS1020
C           TIONS.  A DETAILED DESCRIPTION OF THE GENERAL METHOD AND    NLNS1030
C           PROOF OF CONVERGENCE ARE INCLUDED IN (2).  BASICALLY THE    NLNS1040
C           TECHNIQUE CONSISTS IN EXPANDING THE FIRST EQUATION IN A TAY-NLNS1050
C           LOR SERIES ABOUT THE STARTING GUESS, RETAINING ONLY LINEAR  NLNS1060
C           TERMS, EQUATING TO ZERO AND SOLVING FOR ONE VARIABLE,SAY    NLNS1070
C           X(K), AS A LINEAR COMBINATION OF THE REMAINING N-1 VAR-     NLNS1080
C           IABLES.  IN THE SECOND EQUATION, X(K) IS ELIMINATED BY RE-  NLNS1090
C           PLACING IT WITH ITS LINEAR REPRESENTATION FOUND ABOVE, AND  NLNS1100
C           AGAIN THE PROCESS OF EXPANDING THROUGH LINEAR TERMS,        NLNS1110
C           EQUATING TO ZERO AND SOLVING FOR ONE VARIABLE IN TERMS OF   NLNS1120
C           THE NOW REMAINING N-2 VARIABLES IS PERFORMED.  ONE CONTINUESNLNS1130
C           IN THIS FASHION,ELIMINATING ONE VARIABLE PER EQUATION,      NLNS1140
C           UNTIL FOR THE N'TH EQUATION, WE ARE LEFT WITH ONE EQUATION  NLNS1150
C           IN ONE UNKNOWN.  A SINGLE NEWTON STEP IS NOW PERFORMED, FOL-NLNS1160
C           LOWED BY BACK-SUBSTITUTION IN THE TRIANGULARIZED LINEAR SYS-NLNS1170
C           TEM GENERATED FOR THE X(I)'S.  A PIVOTING EFFECT IS ACHIEVEDNLNS1180
C           BY CHOOSING FOR ELIMINATION AT ANY STEP THAT VARIABLE HAVINGNLNS1190
C           A PARTIAL DERIVATIVE OF LARGEST ABSOLUTE VALUE.  THE PIV-   NLNS1200
C           OTING IS DONE WITHOUT PHYSICAL INTERCHANGE OF ROWS OR COL-  NLNS1210
C           UMNS.                                                       NLNS1220
C                                                                       NLNS1230
C        REFERENCES                                                     NLNS1240
C           1. BROWN, K.M. "ALGORITHM 316 SOLUTION OF SIMULTANEOUS NON- NLNS1250
C              LINEAR EQUATIONS (C5)", COLLECTED ALGORITHMS FROM CACM,  NLNS1260
C              JULY 1967, 316-P 1- 0 AND I- 0.                          NLNS1270
C                           THE REMARKS IMMEDIATELY ABOVE ARE RE-       NLNS1280
C                           PRODUCED FROM THIS WRITE-UP.                NLNS1290
C           2. BROWN, K. M. A QUADRATICALLY CONVERGENT METHOD FOR       NLNS1300
C              SOLVING SIMULTANEOUS NON-LINEAR EQUATIONS.  DOCTORAL     NLNS1310
C              THESIS, DEPT. COMPUTER SCIENCES, PURDUE V., LAFAYETTE,   NLNS1320
C              IND., AUG., 1966.                                        NLNS1330
C           3. BROWN, K. M., AND CONTE, S. D.  THE SOLUTION OF SIM-     NLNS1340
C              ULTANEOUS NONLINEAR EQUATIONS.  PROC. ACM 22ND NAT.      NLNS1350
C              CONF., PP 111-114.                                       NLNS1360
C                                                                       NLNS1370
C     .............................................................     NLNS1380
```

```
C                                                                       NLNS1390
C                                                                       NLNS1400
C                                                                       NLNS1410
C                                                                       NLNS1420


      SUBROUTINE NLNSYS(N,MAXIT,NUMSIG,ISING,IPRINT,EVALUT,X)           NLNS1430
C                                                                       NLNS1440
      DIMENSION X(1)                                                    NLNS1450
      INTEGER CONVRG,TALLY,PCNTER                                       NLNS1460
      COMMON ISUB(99),COE(100,101),PONTER(100,100)
      EXTERNAL EVALUT                                                   NLNS1480
      DIMENSION TEMP(100),PART(100)
    1 CONVRG=1                                                          NLNS1500
C        INITIALIZE CONVERGENCE AND TOLERANCE DETERMINERS.              NLNS1520
      ISING=0                                                           NLNS1530
      RELCON=10.**(-NUMSIG)                                             NLNS1540
C        ITERATION LOOP.                                                NLNS1550
      DO 55 M=1,MAXIT                                                   NLNS1560
C        PROGRAM WILL PRINT OUT SUCCESSIVE APPROXIMATIONS OF X IF       NLNS1570
C        IPRINT IS 0.                                                   NLNS1580
      IF (IPRINT.EQ.0) WRITE (6,1000) M,(X(I),I=1,N)                    NLNS1590
 1000 FORMAT (5X,'ON ITERATION',I4,' THE X ESTIMATE IS',//6(5X,E12.5)/)
C        INITIALIZE FIRST ROW OF ARRAY WHICH ENABLES INDEXING TO PIVOT  NLNS1620
C        THROUGH THE VARIABLES WITHOUT INTERCHANGING ROWS OR COLUMNS.   NLNS1630
      DO 5 J=1,N                                                        NLNS1640
    5 PONTER(1,J)=J                                                     NLNS1650
C        ONE K FOR EACH EQUATION.                                       NLNS1660
      DO 40 K=1,N                                                       NLNS1670
C        IF K IS GREATER THAN 1 BACK-SUBSTITUTION IS NECESSARY.         NLNS1680
      IF (K.GT.1) CALL BAKSUB(K,N,X)                                    NLNS1690
C        SET SIGNAL TO ALLOW INPUT CONSTANTS TO BE READ INTO EVALUT
C        DURING FIRST CALL
      INSIG=1
      IF (M.GT.1) INSIG=0
      CALL EVALUT(X,F,K,INSIG)
C        RESET SIGNAL TO PREVENT EVALUT FROM ATTEMPTING TO READ
C        MORE INPUT.
      INSIG=0
      FACTOR=.001                                                       NLNS1710
    7 TALLY=0                                                           NLNS1720
      DO 10 I=K,N                                                       NLNS1730
      ITEMP=PONTER(K,I)                                                 NLNS1740
      HOLD=X(ITEMP)                                                     NLNS1750
C        GET INCREMENT TO OBTAIN ITEMP'TH PARTIAL.                      NLNS1760
      H=FACTOR*HOLD                                                     NLNS1770
      IF (H.EQ.0.) H=.001                                               NLNS1780
      X(ITEMP)=HOLD+H                                                   NLNS1790
```

```
C         IF K IS GREATER THAN 1, A NEW X(ITEMP) VALUE WILL AFFECT X'S   NLNS1800
C         WHICH ARE EXPRESSED IN TERMS OF IT AS A RESULT OF PREVIOUS     NLNS1810
C         EQUATION STEPS.                                                NLNS1820
      IF (K.GT.1) CALL BAKSUB(K,N,X)                                     NLNS1830
      CALL EVALUT(X,FPLUS,K,INSIG)
C         GET ITEMP'TH PARTIAL.                                          NLNS1850
      PART(ITEMP)=(FPLUS-F)/H                                            NLNS1860
      X(ITEMP)=HOLD                                                      NLNS1870
C         IF PARTIAL IS TOO SMALL, INCREASE TALLY.                       NLNS1880
      IF(ABS(PART(ITEMP)).EQ.0) GO TO 9                                  NLNS1890
      IF(ABS(F/PART(ITEMP)).GT.1.0E20) TALLY=TALLY+1                     NLNS1891
      GO TO 10                                                           NLNS1892
    9 TALLY=TALLY+1                                                      NLNS1893
   10 CONTINUE                                                           NLNS1910
      IF (TALLY.LE.(N-K)) GO TO 15                                       NLNS1920
      FACTOR=FACTOR*10.0                                                 NLNS1930
C         IF SURFACE IS TOO FLAT, THE SINGULARITY INDICATOR IS SET TO    NLNS1940
C         ZERO AND RETURN IS EXECUTED.                                   NLNS1950
      IF (FACTOR.GT..5) GO TO 65                                         NLNS1960
      GO TO 7                                                            NLNS1970
   15 IF(K.LT.N) GO TO 20                                                NLNS1980
C         IF LAST PARTIAL IS ZERO, A SINGULARITY IS INDICATED AND A      NLNS1990
C         RETURN EXECUTED.                                               NLNS2000
      IF (ABS(PART(ITEMP)).EQ.0.) GO TO 65                               NLNS2010
      COE(K,N+1)=0                                                       NLNS2020
      KMAX=ITEMP                                                         NLNS2030
      GO TO 40                                                           NLNS2040
   20 KMAX=PONTER(K,K)                                                   NLNS2050
      DERMAX=ABS(PART(KMAX))                                             NLNS2060
      KPLUS=K+1                                                          NLNS2070
C         GET INDEX FOR LARGEST PARTIAL IN K'TH EQUATION.                NLNS2080
      DO 30 I=KPLUS,N                                                    NLNS2090
      JSUB=PONTER(K,I)                                                   NLNS2100
      TEST=ABS(PART(JSUB))                                               NLNS2110
      IF (TEST.LT.DERMAX) GO TO 25                                       NLNS2120
      DERMAX=TEST                                                        NLNS2130
C         DEFINE PIVOT TO SWIVEL ABOUT THE VARIABLE WITH MAXIMUM PARTIAL NLNS2140
C         WHEN WE GET TO THE NEXT EQUATION.                              NLNS2150
      PONTER(KPLUS,I)=KMAX                                               NLNS2160
C         IF THIS PARTIAL IS BIGGER, WE HAVE A NEW MAXIMUM.              NLNS2170
      KMAX=JSUB                                                          NLNS2180
      GO TO 30                                                           NLNS2190
   25 PONTER(KPLUS,I)=JSUB                                               NLNS2200
   30 CONTINUE                                                           NLNS2210
C         IF THAT PARTIAL IS 0, INDICATE A SINGULARITY AND RETURN.       NLNS2220
      IF (ABS(PART(KMAX)).EQ.0) GO TO 65                                 NLNS2230
      ISUB(K)=KMAX                                                       NLNS2240
      COE(K,N+1)=0                                                       NLNS2250
```

```
      DO 35 J=KPLUS,N
      JSUB=PONTER(KPLUS,J)
C        SAVE THESE CONSTANTS FOR FUTURE USE.
      COE(K,JSUB)=-PART(JSUB)/PART(KMAX)
C        GET PART OF EXPRESSION FOR THE NEW X(KMAX) VALUE
   35 COE(K,N+1)=COE(K,N+1)+PART(JSUB)*X(JSUB)
   40 COE(K,N+1)=(COE(K,N+1)-F)/PART(KMAX)+X(KMAX)
C        IF N IS 1, WE HAVE OUR SOLUTION IN THE NEXT STEP WITHOUT ANY
C        BACK-SUBSTITUTING.
      X(KMAX)=COE(N,N+1)
C        FOR N GREATER THAN 1, WE PERFORM A FINAL BACK-SUBSTITUTION TO
C        GET OUR NEW X-VECTOR.
      IF (N.GT.1) CALL BAKSUB(N,N,X)
      IF (M.EQ.1) GO TO 50
      DO 43 I=1,N
C        TEST FOR CONVERGENCE.
      IF (ABS((TEMP(I)-X(I))/X(I)).GT.RELCON) GO TO 45
   43 CONTINUE
      CONVRG=CONVRG+1
C        IF IT CONVERGES, RETURN WITH LAST VECTOR.
      IF (CONVRG.GE.3) GO TO 60
      GO TO 50
   45 CONVRG=1
C        SAVE CURRENT X-VECTOR FOR TESTING WITH NEXT X-VECTOR.
   50 DO 55 I=1,N
   55 TEMP(I)=X(I)
C        IF M IS THE ITERATION LIMIT, RETURN.
      GO TO 70
   60 MAXIT=M
      GO TO 70
   65 ISING=1
   70 RETURN
      END
C


      SUBROUTINE BAKSUB(K,N,X)
C        THIS SUBROUTINE BACK-SUBSTITUTES OR UPDATES VARIABLES WHICH ARE
C        FUNCTIONS OF CURRENT X ENTRY VALUES.
      DIMENSION X(1)
      INTEGER PONTER
      COMMON ISUB(99),COE(100,101),PONTER(100,100)
C
      DO 10 KMM=2,K
      KM=K+2-KMM
      KMAX=ISUB(KM-1)
      X(KMAX)=0.
      DO 5 J=KM,N
```

```
      JSUB=PONTER(KM,J)                                                 BKSB2720
C         SEE (3) FOR THE EXPRESSION FOR X(KMAX).                       BKSB2730
    5 X(KMAX)=X(KMAX)+COE(KM-1,JSUB)*X(JSUB)                            BKSB2740
   10 X(KMAX)=X(KMAX)+COE(KM-1,N+1)                                     BKSB2750
      RETURN                                                            BKSB2760
      END                                                               BKSB2770
```

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13. ABSTRACT

A finite element method for structural optimization of a prismatic beam of a homogeneous, isotropic material is developed. The beam has a rectangular cross-section of constant fixed height, a fixed length, and a fixed volume. Structural optimum is defined as that shape which allows a maximum load within the elastic range.

A computer program is developed to solve the resulting system of equations and various example problems are solved. Comparison is made with exact optimum beam designs where possible.

The finite element model is able to solve problems with any boundary conditions and types of loading that are consistent with the number of elements selected.

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